

THE PISOT CONJECTURE FOR β -SUBSTITUTIONS

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ABSTRACT. We prove the Pisot Conjecture for β -substitutions: If β is a Pisot number then the tiling dynamical system $(\Omega_{\psi_\beta}, \mathbb{R})$ associated with the β -substitution has pure discrete spectrum. As corollaries: (1) Aritmetical coding of the hyperbolic solenoidal automorphism associated with the companion matrix of the minimal polynomial of any Pisot number is a.e. one-to-one; and (2) All Pisot numbers are weakly finitary.

1. INTRODUCTION.

A *substitution* is a map that takes letters of some finite alphabet to finite words in that alphabet. The space of all bi-infinite words that can arise from infinitely repeating a particular substitution, equipped with the product topology and the shift operation, is called a *substitutive system*. The group \mathbb{R} acts on the suspension (there is a natural choice of roof function) of a substitutive system, resulting in the *tiling dynamical system* associated with the substitution. One would like to know how close the tiling dynamical system is to being simply an action of \mathbb{R} by translation on a compact abelian group. The *Pisot Substitution Conjectures* assert that, under various hypotheses, the tiling dynamical system should be an almost everywhere one-to-one extension of an action of \mathbb{R} by translation on a finite dimensional torus or solenoid.

As long as the substitution is primitive (under some power of the substitution, every letter appears in the image of a single letter), long words are, on average, stretched under substitution by a constant multiplier called the *inflation* of the substitution. The one key hypothesis for the Pisot Substitution Conjectures is that the inflation be a Pisot-Vijayaraghavan (or, simply, Pisot) number - a real algebraic integer greater than 1, all of whose (other) algebraic conjugates lie strictly inside the unit circle. For the tiling dynamical system to have a nontrivial group rotation as a factor it is necessary that the inflation be Pisot ([S2]). The tiling dynamical system of any primitive substitution with Pisot inflation (let's call such a substitution a *Pisot substitution*) is always a finite extension of a translation action on a torus or solenoid ([BKw, BBK]) but maybe not an almost everywhere one-to-one extension of such an action. For example, the tiling dynamical system associated with the Thue-Morse substitution, $a \rightarrow ab, b \rightarrow ba$, with inflation $\beta = 2$, is, at best, an a.e. two-to-one extension of translation on the 2-adic solenoid.

For tiling dynamical systems, equivalent to being an almost everywhere one-to-one extension of an action by rotation on a torus or solenoid is the property of having *pure discrete spectrum* (more on this below). There are two main versions of the Pisot Substitution Conjecture:

(1) If β is Pisot, the tiling dynamical system associated with the β -substitution has pure discrete spectrum.

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(2) The tiling dynamical system of an *irreducible*¹ Pisot substitution has pure discrete spectrum.

Given a real number $\beta > 1$, the β -transformation, $T_\beta : [0, 1] \rightarrow [0, 1]$, is given by $T_\beta(x) = \beta x - \lfloor \beta x \rfloor$, where $\lfloor \cdot \rfloor$ denotes the greatest integer function. If β is Pisot, the orbit of 1 under T_β is finite and so breaks $[0, 1]$ into finitely many subintervals. These subintervals map over each other under T_β in a pattern described by the β -substitution, ψ_β (see Section 2); it is to this substitution that version (1) above refers.

The two Pisot Substitution Conjectures are independent: most irreducible Pisot substitutions don't come from the β -transformation, and most β -substitutions are not irreducible. We prove version (1) of the Pisot Substitution Conjecture in the present article.

In [Ra] G. Rauzy constructed 2-dimensional fractal tiles, called *Rauzy tiles*, that tile the plane both periodically and non-periodically. His construction is based on the 'tribonacci substitution' - the β -substitution associated with the largest root of $x^3 - x^2 - x - 1$. Using arithmetical properties of β -numeration, Thurston ([T]) described a general method for obtaining a self-similar 'shingling' of the plane based on any degree three Pisot unit β . His shingling is a multi-tiling, 'Galois dual' to the tilings of \mathbb{R} provided directly by the β -substitution, in which almost every point in the plane lies in exactly r tiles for some $r \geq 1$. The degree, r , of the shingling is 1 if and only if the shingling is a tiling, and, it turns out, if and only if the tiling dynamical system associated with the β -substitution has pure discrete spectrum ([A2]). Thurston mentions that when these shinglings are, in fact, tilings, they are associated with Markov partitions for hyperbolic toral automorphisms. This procedure for constructing Markov partitions was subsequently developed by Bertrand-Mathis ([B-M]) and Pragastis ([Pr]) and underlies the 'arithmetical coding' of Kenyon, Vershik, Sidorov, Schmidt, and others ([KV, Si1, Si2, Sch2, Ver1, Ver2, Bor] and see Section 3 below).

Solomyak showed in [S1] that a purely arithmetical condition - the so-called Property (F), that every positive element of the ring $\mathbb{Z}[1/\beta]$ have finite β -expansion - is enough to guarantee pure discrete spectrum of the tiling dynamical system associated with the β -substitution and, with Frougny ([FS]), described a collection of polynomials with dominant root a Pisot number satisfying Property (F). Akiyama and Sadahiro ([AS]) and Akiyama ([A1]) showed that Thurston's shingling produces a tiling when the Pisot unit β satisfies Property (F). In [H] Hollander introduced the weaker Property (W) (see Section 3 below), which Akiyama ([A2]) subsequently proved to be equivalent to pure discrete spectrum for the 1-d tiling system (and to the Galois dual shingling being a tiling). In [Si1], Sidorov proves that, for Pisot units, Property (W) is equivalent to arithmetical coding (with fundamental homoclinic point) being a.e. 1-1.

Interest in nonperiodic tilings and spectral properties of substitutions was stimulated by Schectman's discovery, in the early 80's, of quasicrystalline materials. Self-similar point sets that have good diffractive properties, and so provide mathematical models of quasicrystals, can be created with substitutions of Pisot type: if one places an 'atom' at a characteristic position in each tile of a Pisot substitution tiling (of whatever dimension) the diffraction of the resulting arrangement will have Bragg peaks. In fact, the diffraction spectrum of the arrangement will be pure point (as for a perfect quasicrystal) if and only if the tiling dynamical system has pure discrete spectrum, meaning that the eigenfunctions of the dynamical system span the space L^2 of square-integrable

¹ A substitution is irreducible if the characteristic polynomial of its abelianization (or substitution matrix - see Section 5) is irreducible over the rationals.

functions on the tiling space ([D],[LMS], and see [Le] for a survey). For substitution tiling dynamical systems, it is a consequence of the Halmos- von Neumann theory (and Solomyak's result that eigenfunctions are continuous ([S4])) that pure discrete spectrum of the tiling dynamical system is equivalent to the system being an almost everywhere one-to-one extension of a translation action on a torus or solenoid (see [BK] for more detail and related characterizations).

For more on Pisot substitutions and the Pisot Substitution Conjectures, see the surveys [BS] and [ABBLS]. A general introduction to the dynamical and topological properties of substitution tiling spaces can be found in [S2], [Ro], and [AP]. The direct antecedent to the current article is [B1] in which the author dealt with β -substitutions for β a Pisot simple Parry number. As outlined below, we extend the approach taken there to all Pisot numbers. In the summary that follows, θ is an arbitrary (primitive, nonperiodic) substitution with Pisot inflation β , ψ_β is the β -substitution, and Θ denotes the substitution-induced homeomorphism of the substitution tiling space Ω_θ . Here are the key steps in our approach:

- (1) There is an almost everywhere cr -to-1 map $\pi_{max} : \Omega_\theta \rightarrow \hat{\mathbb{T}}^d$ that factors the \mathbb{R} -action on Ω_θ onto a Kronecker action on a d -dimensional solenoid $\hat{\mathbb{T}}^d$. Here d is the algebraic degree of β and $cr = cr(\theta) < \infty$ is the coincidence rank of the substitution θ .²
- (2) The system $(\Omega_\theta, \mathbb{R})$ has pure discrete spectrum if and only if $cr(\theta) = 1$.
- (3) The structure relation for π_{max} is strong regional proximality (\sim_{srp}): For $T, T' \in \Omega_\theta$, $\pi_{max}(T) = \pi_{max}(T')$ if and only if $T \sim_{srp} T'$. Furthermore, there are $T_1, \dots, T_r \in \Omega_\theta$ with $T_i \sim_{srp} T_j$ and $T_i \cap T_j = \emptyset$ for $i \neq j$ if and only if $r \leq cr(\theta)$.
- (4) The relation \sim_s , defined on Ω_θ by $T \sim_s T'$ if $d(\Theta^k(T), \Theta^k(T')) \rightarrow 0$ as $k \rightarrow \infty$ is open in the sense that $T \sim_s T' \implies T - t \sim_s T' - t$ for $|t|$ small.
- (5) If τ and τ' are tiles, let $\tau \sim_s \tau'$ mean $T \sim_s T'$ for all $T, T' \in \Omega_\theta$ with $\tau \in T, \tau' \in T'$. Then $cr(\theta) = 1$ if and only if there is a tile τ (equivalently, for all tiles τ) and there are $t_i \neq 0$ with $t_i \rightarrow 0$ as $i \rightarrow \infty$ so that, for each i , $\tau - t_i - t \sim_s \tau - t$ for a dense set of $t \in spt(\tau) \cap spt(\tau - t_i)$.
- (6) The stable equivalence relation (\approx_s) is defined on a tiling space Ω_θ by $T \approx_s T'$ if and only if $T \sim_{srp} T'$ and $T - t \sim_s T' - t$ for a set of t dense in \mathbb{R} . Then $cr(\theta) = 1$ if and only if $\approx_s = \sim_{srp}$. If $cr(\theta) > 1$, then $(\Omega_\theta / \approx_s) \simeq \Omega_{\tilde{\theta}}$ is a substitution tiling space for a substitution $\tilde{\theta}$, $cr(\tilde{\theta}) = cr(\theta)$, and \approx_s is trivial on $\Omega_{\tilde{\theta}}$.
- (7) There is some leeway in choosing a substitution to generate an isomorphism class of tiling spaces. If $cr(\psi_\beta) > 1$, the substitution $\tilde{\psi}_\beta$ may be chosen to have the special property: there is an $n \in \mathbb{N}$ and a letter b in the alphabet $\mathcal{A}_{\tilde{\psi}_\beta}$ for $\tilde{\psi}_\beta$ so that $\tilde{\psi}_\beta^n(a) = b \cdots a$ for all $a \in \mathcal{A}_{\tilde{\psi}_\beta}$.
- (8) If θ is any substitution for which there are $n \in \mathbb{N}$ and $b \in \mathcal{A}_\theta$ with $\theta^n(a) = b \cdots a$ for all $a \in \mathcal{A}_\theta$, then \approx_s is nontrivial on Ω_θ .

Pure discrete spectrum for $(\Omega_{\psi_\beta}, \mathbb{R})$ then follows: If the spectrum were not pure discrete, the substitution $\tilde{\psi}_\beta$ would be such that \approx_s is trivial on $\Omega_{\tilde{\psi}_\beta}$ (by (6)). But this is contradicted by (7) and (8). Versions of items (1)-(6) hold quite generally for substitutions of Pisot type in any dimension. For (1)-(3) see [BKw, BBK, BK, B2], (4) comes from [BO], a version of (5) first appears in [BBK] and in the form stated here in [B1] with a proof derived from [BSW], and (6) is from [B1]. Item (8) is from [B1]. Section 4 is devoted to proving item (7) for the particular case of β -substitutions. The

²The coincidence rank is equal to the degree of the 'Galois dual' shingling, or multi-tiling, of \mathbb{R}^{d-1} mentioned previously.

argument is somewhat technical and is based on the topological structure of Ω_{ψ_β} , as we describe now.

There are tilings in Ω_{ψ_β} , corresponding to the fixed point 0 of T_β and to the point of the periodic orbit, of period p , of T_β on which the T_β -orbit of 1 eventually lands. These tilings are periodic under Ψ_β of period p and pairwise strong regionally proximal. Using the basic technique of [BD2], and the special ‘monotonic’ nature of ψ_β , we show that these tilings are entangled: for T, T' any two of them $T - t \sim_s T' - t$ for a set of t dense in \mathbb{R}^+ . But also, for such $T \neq T'$, we show that T and T' are not asymptotic on \mathbb{R}^+ (there are arbitrarily large t so that the tiles of T and T' at t are distinct). From this it follows (again using the monotonicity of ψ_β) that there is $\epsilon > 0$ so that if τ and τ' are any two tiles having the same initial vertex x , then $\tau - t \sim_s \tau' - t$ for a set of t dense in $[x, x + \epsilon)$, and there are tiles $\tau_-^1, \dots, \tau_-^p$ so that if τ is any tile, there is exactly one of the τ_-^i so that, with t_0 chosen such that the terminal vertices of $\tau - t_0$ and τ_-^i are both equal to y , $\tau - t_0 - t \sim_s \tau_-^i - t$ for a set of t dense in $(y - \epsilon, y]$. Item (7) then follows.

As mentioned earlier, Akiyama (in [A2]) established the equivalence of the arithmetical Property (W) with pure discrete spectrum of $(\Omega_{\psi_\beta}, \mathbb{R})$, and Sidorov (in [Si1]) proved the equivalence of Property (W) with a.e.1-1-ness of arithmetical coding (at least for Pisot units). In the final Section 5 we reprove these results (and extend the Sidorov result to non-units) from a unified point of view. We think it is of interest to note that arithmetical coding occurs in the context of hyperbolic dynamics, while the map onto the maximal equicontinuous factor is determined by properties of the translation action. We will tie these two viewpoints together by showing the structure relation for the maximal equicontinuous factor map (strong regional proximality) is the same as the ‘global shadowing’ relation defined in terms of the hyperbolic homeomorphism Ψ_β (such an equivalence holds also in higher dimensions - see [BG]).

Finally, Property (W) will be interpreted as a statement about homoclinic return times. The relation between the structure of these return times and pure discreteness of the translation action is a general phenomenon (see, for example, Corollary 4.5 of [B2]).

2. BACKGROUND ON β -NUMERATION AND SUBSTITUTION TILINGS.

For $\beta > 1$, the β -transformation, $T_\beta : [0, 1] \rightarrow [0, 1]$ is given by

$$T_\beta(x) := \beta x - \lfloor \beta x \rfloor,$$

where $\lfloor \cdot \rfloor$ denotes the greatest integer function. If β is a Pisot number, then the T_β -orbit of 1, $\mathcal{O}_{T_\beta}(1) := \{T_\beta^n(1) : n \geq 0\}$, is finite ([B-M], [Sch1]). Such a β is a *simple Parry number* if $0 \in \mathcal{O}_{T_\beta}(1)$ and otherwise is a *non-simple Parry number*. Every $x \in [0, 1)$ has an *itinerary*

$$\kappa(x) = x_1 x_2 \cdots,$$

with $x_n := i \in \{0, \dots, \lfloor \beta \rfloor\}$ provided $T_\beta^{n-1}(x) \in [i/\beta, (i+1)/\beta)$. For such x

$$x = \sum_{n=1}^{\infty} x_n \beta^{-n}$$

is the (*greedy*) β -expansion of x . The *kneading invariant* or *characteristic sequence* of β is given by

$$\kappa(1) = c_1 c_2 \cdots := \lim_{x \nearrow 1} \kappa(x),$$

the limit being taken in the product topology on the sequences. If β is a simple Parry number, then $\kappa(1)$ is periodic of some (least) period p : $\kappa(1) = \overline{c_1 \cdots c_p}$. If β is a non-simple Parry number, then

$\kappa(1)$ is strictly preperiodic: there are $m \geq 1$ and $p \geq 1$ with $\kappa(1) = c_1 \cdots c_m \overline{c_{m+1} \cdots c_{m+p}}$, where we take m , and then p , as small as possible.

The *one-sided β -shift* is the Cantor dynamical system

$$(X_\beta^+, \sigma),$$

with $X_\beta^+ := cl(\{(x_1, x_2, \dots) : x_1 x_2 \cdots = \kappa(x) \text{ for some } x \in [0, 1]\})$ and $\sigma((x_1, x_2, x_3, \dots)) := (x_2, x_3, \dots)$. One has that X_β consists precisely of the sequences of digits from $\{0, \dots, \lfloor \beta \rfloor\}$ all of whose shifts are lexicographically no larger than the sequence (c_1, c_2, \dots) corresponding to $\kappa(1)$ ([Pa]). The (two-sided) β -shift is obtained by taking inverse limits: $X_\beta := \{(\dots, x_{-1}, x_0, x_1, \dots) : (x_k, x_{k+1}, \dots) \in X_\beta^+ \text{ for all } k \in \mathbb{Z}\}$. For each $x \in \mathbb{R}^+$ there is a corresponding point

$$\underline{x} := (\dots, 0, 0, x_{-k}, \dots, x_{-1}x_0, x_1, \dots) \in X_\beta$$

where $x = \sum_{i=-k}^\infty x_i \beta^{-i}$, and $\sum_{i=1}^\infty x_{-k+i} \beta^{-i}$ is the greedy β -expansion of $\beta^{-k}x \in [0, 1)$. For more detail on the β -shift, see [R, Pa, IT, Bl, LM].

For a given Pisot β , let $\{0, 1\} \cup \{T_\beta^i(1) : i = 1, \dots, m+p\} = \{0 = s_0 < s_1 < \dots < s_{m+p} = 1\}$. The *prototiles* are the marked intervals $\tau_i := [s_{i-1}, s_i] \times \{i\}$, $i = 1, \dots, m+p$; their *vertices* are $\min(\tau_i) := s_{i-1}$ and $\max(\tau_i) := s_i$. The *support* of τ_i is $spt(\tau_i) := [s_{i-1}, s_i]$. We call $\tau = \tau_i + t := (spt(\tau_i) + t) \times \{i\}$ a *tile of type i* with support $spt(\tau_i) + t$ and vertices $\min(\tau) := s_{i-1} + t$ and $\max(\tau) := s_i + t$. The β -*substitution* on the alphabet $\mathcal{A} = \{1, \dots, m+p\}$ is the map $\psi_\beta : \mathcal{A} \rightarrow \mathcal{A}^*$, \mathcal{A}^* being the collection of finite nonempty words on \mathcal{A} , given by $\psi_\beta(i) = i_1 i_2 \cdots i_n$ provided $T_\beta(x)$ passes through the support of τ_{i_1} , then the support of τ_{i_2}, \dots , then the support of τ_{i_n} as x increases from s_{i-1} to s_i .³ The corresponding *tile substitution* Ψ_β is defined on prototiles by $\Psi_\beta(\tau_i) := \{\tau_{i_1} - \min(\tau_{i_1}) + \beta \min(\tau_i), \tau_{i_2} - \min(\tau_{i_2}) + \max(\tau_{i_1}) - \min(\tau_{i_1}) + \beta \min(\tau_i), \dots, \tau_{i_n} - \min(\tau_{i_n}) + (\sum_{j=1}^{n-1} (\max(\tau_{i_j}) - \min(\tau_{i_j}))) + \beta \min(\tau_i)\}$. (That just says that Ψ_β multiplies the support of τ_i by a factor of β and tiles the resulting interval by the τ_{i_j} , following the pattern of ψ_β .) The tile substitution Ψ_β is then defined on arbitrary tiles by $\Psi_\beta(\tau_i + t) := \Psi_\beta(\tau_i) + \beta t$. By a *patch P* we will mean a finite collection of tiles with the properties: if $\tau \neq \tau' \in P$, then $\text{int}(spt(\tau)) \cap \text{int}(spt(\tau')) = \emptyset$; and $spt(P) := \cup_{\tau \in P} spt(\tau)$ is connected. Ψ_β extends to patches by $\Psi_\beta(P) := \cup_{\tau \in P} \Psi_\beta(\tau)$.

A *tiling T* of \mathbb{R} is a collection of tiles with the properties: if $\tau \neq \tau' \in T$ then $\text{int}(spt(\tau)) \cap \text{int}(spt(\tau')) = \emptyset$; and $\cup_{\tau \in T} spt(\tau) = \mathbb{R}$. A patch P is *allowed* for Ψ_β if there is a tile $\tau = \tau_i + t$ and an $n \in \mathbb{N}$ so that $P \subset \Psi_\beta^n(\tau)$. For $\beta \notin \mathbb{N}$, the *tiling space* associated with Ψ_β is the collection of tilings of \mathbb{R} , all of whose patches are allowed for Ψ_β :

$$\Omega_{\psi_\beta} := \{T : \text{every patch } P \subset T \text{ is allowed for } \Psi_\beta\}.$$

For $r \geq 0$ and $T \in \Omega_{\psi_\beta}$, let

$$B_r[T] := \{\tau \in T : spt(\tau) \cap [-r, r] \neq \emptyset\}.$$

So, for example, $B_0[T]$ is the collection of tiles of T whose supports contain the origin.

The group \mathbb{R} acts on Ω_{ψ_β} by translation, $T - t := \{\tau - t : \tau \in T\}$, and there is a metric d (called the *tiling metric*) on Ω_{ψ_β} with the property that $d(T, T')$ is small if there are r, t, t' with r large and $|t|, |t'|$ small, so that $B_r[T - t] = B_r[T' - t']$ (see, for example, [AP]). With the topology induced by d , Ω_{ψ_β} is compact and the translation action on Ω_{ψ_β} is continuous. It is well-known that if $\beta \notin \mathbb{N}$,

³Typically, the prototiles are taken to be $[0, s_i] \times \{i\}$, resulting in a different ' β -substitution'. It is straightforward to pass from one to the other by a rewriting procedure (see [D] or [BD1]) and the resulting tiling dynamical systems are isomorphic.

then ψ_β is *non-periodic* (meaning $T - t = T$ for some $T \in \Omega_{\psi_\beta}$ implies $t = 0$) and *primitive* (there are n and i so that all letters occur in $\psi_\beta^n(i)$) and it follows that the translation dynamical system $(\Omega_{\psi_\beta}, \mathbb{R})$ is minimal and uniquely ergodic ([S2]). We will denote the unique Borel translation invariant measure on Ω_{ψ_β} by μ . Moreover, $\Psi_\beta : \Omega_{\psi_\beta} \rightarrow \Omega_{\psi_\beta}$ by $\Psi_\beta(T) := \cup_{\tau \in T} \Psi_\beta(\tau)$ is a homeomorphism with μ invariant and ergodic ([S3]). Note that the translation- and Ψ_β - dynamics are intertwined by:

$$\Psi_\beta(T - t) = \Psi_\beta(T) - \beta t.$$

If $\beta = n$ is an integer, the substitution ψ_β is the periodic substitution $1 \mapsto 1^n$ and, as usually defined, the tiling space is the circle \mathbb{T}^1 and the Ψ_n is the n -fold covering map $x + \mathbb{Z} \mapsto nx + \mathbb{Z}$. It will be convenient (looking ahead to the discussion of arithmetical coding) to instead define Ω_{ψ_n} to be the solenoid $\varprojlim \Psi_n$, interpret Ψ_n to be the shift homeomorphism, and take the \mathbb{R} -action to be the natural Kronecker action $(x_1 + \mathbb{Z}, x_2 + \mathbb{Z}, \dots) - t := (x_1 - t + \mathbb{Z}, x_2 - t/n + \mathbb{Z}, \dots)$.

A continuous map $f : \Omega_{\psi_\beta} \rightarrow \mathbb{T}^1 := \mathbb{R}/\mathbb{Z}$ is a *continuous eigenfunction* of $(\Omega_{\psi_\beta}, \mathbb{R})$ if there is $\gamma \in \mathbb{R}$ (the associated eigenvalue) so that $f(T - t) = f(T) - (\gamma t + \mathbb{Z})$ for all $t \in \mathbb{R}$. The system $(\Omega_{\psi_\beta}, \mathbb{R})$ is said to have *pure discrete spectrum* if the continuous eigenfunctions span $L^2(\mu)$. An alternative formulation of this property, in terms of equicontinuous factors, is closer to the spirit of this article. A continuous system (Y, \mathbb{R}) (that is, a continuous action of \mathbb{R} on the metric space Y) is *equicontinuous* provided the collection of homeomorphisms $\{y \mapsto y \cdot t\}_{t \in \mathbb{R}}$ is equicontinuous. A *maximal equicontinuous factor* of a system (X, \mathbb{R}) is an equicontinuous factor (X_{max}, \mathbb{R}) of (X, \mathbb{R}) with the property that every equicontinuous factor of (X, \mathbb{R}) is also a factor of (X_{max}, \mathbb{R}) . Compact minimal systems always have a (unique up to isomorphism) maximal equicontinuous factor ([Aus]). In the case at hand, the maximal equicontinuous factor of $(\Omega_{\psi_\beta}, \mathbb{R})$ is a Kronecker action on a d -dimensional torus or solenoid $\hat{\mathbb{T}}_\beta^d$, with d the algebraic degree of β ([BBK],[B2]). Here $\hat{\mathbb{T}}_\beta^d$ is the inverse limit $\hat{\mathbb{T}}_\beta^d := \varprojlim F_M$ with M the companion matrix of the minimal polynomial of β and $F_M : \mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d \rightarrow \mathbb{T}^d$ the hyperbolic toral endomorphism $F_M(v + \mathbb{Z}^d) := Mv + \mathbb{Z}^d$ and the Kronecker action is given by $(v_1 + \mathbb{Z}^d, v_2 + \mathbb{Z}^d, v_3 + \mathbb{Z}^d, \dots) - t := (v_1 - t\omega + \mathbb{Z}^d, v_2 - \beta^{-1}t\omega + \mathbb{Z}^d, v_3 - \beta^{-2}t\omega + \mathbb{Z}^d, \dots)$, where ω is a right positive eigenvector for the eigenvalue β of M .

The structure relation for the maximal equicontinuous factor map $\pi_{max} : \Omega_{\psi_\beta} \rightarrow \hat{\mathbb{T}}_\beta^d$ is given by strong regional proximality ⁴ ([BK]): Tilings T, T' are *strong regionally proximal*, denoted $T \sim_{srp} T'$, provided for all $R > 0$ there are $S_R, S'_R \in \Omega_{\psi_\beta}$ and $t_R \in \mathbb{R}$ so that

$$\begin{aligned} B_R[T] &= B_r[S_R], \\ B_R[T'] &= B_r[S'_R], \text{ and} \\ B_R[S_R - t_R] &= B_r[S'_R - t_R]. \end{aligned}$$

There is $r \in \mathbb{N}$, called the *coincidence rank* of ψ_β so that $\pi_{max} : \Omega_{\psi_\beta} \rightarrow \hat{\mathbb{T}}_\beta^d$ is a.e. r -to-1 and for each $z \in \hat{\mathbb{T}}_\beta^d$ there are $S_1, \dots, S_r \in \pi_{max}^{-1}(z)$ with $S_i \cap S_j = \emptyset$ for $i \neq j$. The system $(\Omega_{\psi_\beta}, \mathbb{R})$ has pure discrete spectrum if and only if the coincidence rank of ψ_β equals 1, that is, if and only if π_{max} is a.e. 1-1 (see [BKw, BBK, BK]).

Conjecture 1. (*Pisot conjecture for β -substitutions*) *If β is a Pisot number, then $(\Omega_{\psi_\beta}, \mathbb{R})$ has pure discrete spectrum.*

⁴In the general context of an abelian group acting minimally on a compact metric space, the structure relation for the maximal equicontinuous factor map is regional proximality ([V]). For Pisot substitution tiling spaces, the Meyer property is responsible for the strong version we use here.

3. ARITHMETICAL CODING AND PROPERTY (W)

Given a Pisot number β of algebraic degree d and M the companion matrix of its minimal polynomial, let E^u and E^s denote the 1-dimensional and $(d-1)$ -dimensional unstable and stable spaces of M . (So E^u is spanned by the positive right eigenvector ω and $E^s = \{v \in \mathbb{R}^d : M^n v \rightarrow 0 \text{ as } n \rightarrow \infty\}$.) Then $\mathbb{R}^d = E^u \oplus E^s$ and for $y \in \mathbb{R}^d$ we write $y = y^u + y^s$ with $y^u \in E^u$ and $y^s \in E^s$. A homoclinic point $\bar{y} = y + \mathbb{Z}^d \in \mathbb{T}^d$, by which we mean $y = z^u$ for some $z \in \mathbb{Z}^d$, is *fundamental* if $\{z, Mz, \dots, M^{d-1}z\}$ is a basis for \mathbb{Z}^d . (Such a point always exists.) If \bar{y} is a fundamental homoclinic point, let $\hat{\bar{y}} := (y + \mathbb{Z}^d, M^{-1}y + \mathbb{Z}^d, \dots) \in \hat{\mathbb{T}}_\beta^d$ be the corresponding point, homoclinic to $\hat{0}$ under the shift automorphism \hat{F}_M . The map

$$h_{\bar{y}} : X_\beta \rightarrow \hat{\mathbb{T}}_\beta^d$$

given by

$$h_{\bar{y}}((x_i)) := \sum_{i \in \mathbb{Z}} x_i \hat{F}_M^{-i}(\hat{\bar{y}})$$

is called an *arithmetical coding* of the hyperbolic automorphism $\hat{F}_M : \hat{\mathbb{T}}_\beta^d \rightarrow \hat{\mathbb{T}}_\beta^d$. Such a coding has the felicitous properties:

$$h_{\bar{y}}(\beta x) = \hat{F}_M(h_{\bar{y}}(\underline{x}))$$

for all $x \in \mathbb{R}^+$, more generally

$$h_{\bar{y}} \circ \sigma = \hat{F}_M \circ h_{\bar{y}},$$

and

$$h_{\bar{y}}(\underline{x + x'}) = h_{\bar{y}}(\underline{x}) + h_{\bar{y}}(\underline{x'})$$

for all $x, x' \in \mathbb{R}^+$.

The point $x \in \mathbb{R}^+$ is said to have *finite β -expansion* if $\underline{x} = (\dots, 0, x_{-k}, \dots, x_n, 0, 0, \dots)$ for some $k \in \mathbb{N}$. Let $Fin(\beta) := \{x \in \mathbb{R}^+ : x \text{ has finite } \beta\text{-expansion}\}$. The Pisot number β is said to be *finitary* if $\mathbb{Z}[1/\beta] \cap \mathbb{R}^+ \subset Fin(\beta)$ and is said to be *weakly finitary* if for all $z \in \mathbb{Z}[1/\beta] \cap \mathbb{R}^+$ there are $x, y \in Fin(\beta)$, with y as small as desired, such that $z = x - y$. The so-called *Property (W)* is that β is weakly finitary.

Akiyama proves in [A1] that β satisfies Property (W) if and only if $(\Omega_{\psi_\beta}, \mathbb{R})$ has pure discrete spectrum and Sidorov ([Si1]) shows that, at least for Pisot β that are algebraic units, Property (W) is equivalent to arithmetical coding (with fundamental homoclinic point) being a.e. 1-1. It is shown in [ARS] that all cubic Pisot units, and higher degree Pisot numbers satisfying a ‘dominant condition’ are weakly finitary. We prove Conjecture 1 in the next section and provide an alternative argument for the equivalence of Conjecture 1 with a.e. one-to-oneness of arithmetical coding for Pisot β and satisfaction of Property (W) for all Pisot β in the final section. (In fact we will prove the stronger form of Property (W): If β is Pisot and $z \in \mathbb{Z}[1/\beta] \cap \mathbb{R}^+$, then $\{y : y \in Fin(\beta) \text{ and } y+z \in Fin(\beta)\}$ is dense in \mathbb{R}^+ .)

4. PROOF OF THE PISOT CONJECTURE FOR β -SUBSTITUTIONS.

The *language* of ψ_β is defined by $\mathcal{L}(\psi_\beta) := \{w \in \mathcal{A}^* : w \text{ is a factor of } \psi_\beta^n(i) \text{ for some } n \in \mathbb{N}, i \in \mathcal{A}\}$.

The following ‘monotonicity’ properties of $\mathcal{L}(\psi_\beta)$ will be used repeatedly.

Property 1: If ab is a two letter word in $\mathcal{L}(\psi_\beta)$ and $b \in \{2, \dots, m+p\}$, then $a = b-1$.

Property 2: If ab and ac are two letter words in $\mathcal{L}(\psi_\beta)$ and $b \neq c$, then either $b = 1$ or $c = 1$.

Property 3: If ac and bc are two letter words in $\mathcal{L}(\psi_\beta)$ and $a \neq b$, then $c = 1$.

The first property is a simple consequence of the fact that T_β increases monotonically from 0 on each of its maximal intervals of continuity. Properties 2 and 3 follow immediately from Property 1. Note that Property 1 implies that if $\tau_k + t \in T \in \Omega_{\psi_\beta}$, then $\{\tau_1, \tau_2, \dots, \tau_k\} + t \subset T$.

We have labeled the prototiles according to the order of their occurrence on the interval $[0, 1]$. It will be convenient to also have labelings of the prototiles according to where, along the orbit of 1 under T_β , their maximum and minimum vertices lie: Let $z^i := T_\beta^{i-1}(1)$ for $i = 1, \dots, m + p$; then if $\max(\tau_i) = z^j$, define $\tau_-^j := \tau_i$, and if $\min(\tau_i) = z^j$ set $\tau_+^j := \tau_i$. If $m > 0$, then corresponding to each of the T_β -periodic points z^{m+i} , $i = 1, \dots, p$, there are two Ψ_β -periodic tilings:

$$T_i := \cup_{n=0}^{\infty} \Psi_\beta^{np}(\{\tau_-^{m+i} - z^{m+i}, \tau_+^{m+i} - z^{m+i}\})$$

and

$$T_i^0 := \cup_{n=0}^{\infty} \Psi_\beta^{np}(\{\tau_-^{m+i} - z^{m+i}, \tau_1\}).$$

Note that $T_j^0 \sim_{srp} T_i^0$ and $T_k^0 \sim_{srp} T_k$ for all $i, j, k \in \{1, \dots, p\}$, so also $T_i \sim_{srp} T_j$ for all $i, j \in \{1, \dots, p\}$.

Given $T, T' \in \Omega_{\psi_\beta}$ we write $T \sim_s T'$ provided $d(\Psi_\beta^k(T), \Psi_\beta^k(T')) \rightarrow 0$ as $k \rightarrow \infty$. Since $d(\Psi_\beta^k(T), \Psi_\beta^k(T')) \rightarrow 0$ as $k \rightarrow \infty$ if and only if there is $k \in \mathbb{N}$ so that $B_0[\Psi_\beta^k(T)] = B_0[\Psi_\beta^k(T')]$ ([BO]), we see that $T \sim_s T' \implies T - t \sim_s T' - t$ for all $|t| < \epsilon$, for some $\epsilon > 0$. We'll say that $T \sim_s T'$ *densely on the interval J* if $T - t \sim_s T' - t$ for a dense set of $t \in J$. Note that then $T - t \sim_s T' - t$ for an open dense subset of t in J . If P, P' are allowed patches, we'll say that $P \sim_s P'$ *densely on J* if $T \sim_s T'$ densely on J for all $T, T' \in \Omega_{\psi_\beta}$ with $P \subset T, P' \subset T'$.

Lemma 2. *Suppose that $m > 0$ and $p > 1$. There are then $i \neq j$ so that $T_i \sim_s T_j$ densely on \mathbb{R}^+ . If $m > 0$ and $p = 1$, then $T_1 \sim_s T_1^0$ densely on \mathbb{R}^+ .*

Proof. We give a variant of the basic argument of [BD2]. Let r be the coincidence rank of ψ_β . We may assume that $r > 1$, otherwise $T_i \sim_s T_j \sim_s T_k^0$ densely on \mathbb{R} for all i, j, k . ($T_i \sim_{srp} T_j \sim_{srp} T_k^0$ for all i, j, k , as noted above, and, in the context of any Pisot substitution tiling, $T \sim_{srp} T' \implies T \sim_s T'$ densely on \mathbb{R} - see [BK].) There are tilings S_1, \dots, S_r with the properties (see [BK]):

- (1) $S_i \sim_{srp} S_j$ for $1 \leq i, j \leq r$,
- (2) S_i is Ψ_β -periodic for each $i \in \{1, \dots, r\}$, and
- (3) $S_i \cap S_j = \emptyset$ for all $i \neq j \in \{1, \dots, r\}$.

Replacing Ψ_β by an appropriate power, we may assume that the S_i are all fixed by Ψ_β . For each i , let V_i be the collection of all the vertices of tiles in S_i and let $V = \{\dots v_{-1} < v_1 < v_2 \dots\} := \cup_{i=1}^r V_i$. For each $j \in \mathbb{Z}$ and $i \in \{1, \dots, r\}$, let η_j^i be the tile of S_i with $(v_j + v_{j+1})/2 \in \text{spt}(\eta_j^i)$. We call the collection $\mathcal{C}_j := \{\eta_j^i : i = 1, \dots, r\}$ a *configuration*. For each $j \in \mathbb{Z}$, let $m_j := \max\{\min(\eta_j^i) : i = 1, \dots, r\}$ and $M_j := \min\{\max(\eta_j^i) : i = 1, \dots, r\}$.

Up to translation, there are only finitely many configurations (this is the Meyer property, see [BK]). Thus, since the S_i are not translation-periodic, there are $k, l \in \mathbb{Z}$ and $w \in \mathbb{R}$ with $\mathcal{C}_k = \mathcal{C}_l + w$ but $\mathcal{C}_{k+1} \neq \mathcal{C}_{l+1} + w$. For each $i \in \{1, \dots, r\}$, let $i' \in \{1, \dots, r\}$ be (uniquely) defined by $\eta_k^{i'} = \eta_l^i + w$. Note that if $\max(\eta_j^i) > M_j$, then $\eta_{j+1}^i = \eta_j^i$. Hence, letting $\mathcal{C}_l' := \{\eta_l^i : \eta_{k+1}^{i'} \neq \eta_{l+1}^i\}$ we see that $\max(\eta_l^i) = M_l^i$ for all $\eta_l^i \in \mathcal{C}_l'$. Furthermore, if $\eta_l^i \in \mathcal{C}_l'$, then either η_{l+1}^i is of type

1 (that is, is a translate of τ_1) or $\eta_{k+1}^{i'}$ is of type 1 (this is Property 2). Since the S_i are pairwise disjoint, it follows that $\sharp C'_l \in \{1, 2\}$.

We consider two cases:

Case 1: $\sharp C'_l = 1$.

Let $\{\eta_l^i\} = C'_l$. We claim that $\eta_{l+1}^i + w \sim_s \eta_{k+1}^{i'}$ densely on $[0, \epsilon)$, where $\epsilon := \min\{M_{l+1} - m_{l+1}, M_{k+1} - m_{k+1}\}$. If not, there is $t_0 \in (0, \epsilon)$ and $\delta > 0$ so that $B_0[\Psi_\beta^n(S_i - t)] \cap B_0[\Psi_\beta^n(S_{i'} - w - t)] = \emptyset$ for all $n \in \mathbb{N}$ and $|t - t_0| < \delta$. That is, $B_{\beta^n \delta/2}[\Psi_\beta^n(S_i - t_0)] \cap B_{\beta^n \delta/2}[\Psi_\beta^n(S_{i'} - w - t_0)] = \emptyset$ for all $n \in \mathbb{N}$. Choose $n_k \rightarrow \infty$ so that $\Psi_\beta^{n_k}(S_j - t_0) \rightarrow W_j \in \Omega_{\psi_\beta}$ and $\Psi_\beta^{n_k}(S_j + w - t_0) \rightarrow U_j \in \Omega_{\psi_\beta}$ as $k \rightarrow \infty$, for all $j = 1, \dots, r$ (such $\{n_k\}$ exists by compactness of Ω_{ψ_β}). Then the W_j , $j = 1, \dots, r$, are pairwise disjoint (the S_j are pairwise disjoint and fixed by Ψ_β) and strongly regionally proximal (\sim_{srp} is closed and preserved by both translation and Ψ_β), as are the U_j . Furthermore, $W_i \cap U_{i'} = \emptyset$. Since $\emptyset \neq \{W_j : j \neq i\} = \{U_j : j \neq i'\}$, $W_i \sim_{srp} U_{i'}$. We now have $r + 1$ pairwise disjoint and strongly regionally proximal tilings $(W_1, \dots, W_r, U_{i'})$, contradicting coincidence rank r .

So $\eta_{l+1}^i + w \sim_s \eta_{k+1}^{i'}$ densely on $[0, \epsilon)$. Again, since $\eta_l^i + w = \eta_k^{i'}$, it follows from Property 2 that one of η_{l+1}^i and $\eta_{k+1}^{i'}$ must be of type 1: say $\eta_{l+1}^i = \tau_1 + M_l$ and $\eta_{k+1}^{i'} = \tau_j - \min(\tau_j) + M_l + w$. We have that $\tau_1 \sim_s \tau_j - \min(\tau_j)$ densely on $[0, \epsilon)$ (and $j \in \{2, \dots, m + p\}$ since $C_{k+1} \neq C_{l+1} + w$). Let $q \in \{1, \dots, p\}$ be such that q and j are in phase, that is, $p \mid (q - j)$. Then $\tau_1 \sim_s \tau_q - \min(\tau_q)$ densely on $[0, \epsilon)$. Since $\Psi_\beta^p(\tau_1) = \{\tau_1, \dots\}$ and $\Psi_\beta^p(\tau_q - \min(\tau_q)) = \{\tau_q - \min(\tau_q), \dots\}$ we see that $T_n^0 \sim_s T_q$ densely on \mathbb{R}^+ (for all n). Then $T_j = \Psi_\beta^{j-q}(T_q) \sim_s \Psi_\beta^{j-q}(T_n^0) = T_{n+j-q}^0 \sim_s T_q$ densely on \mathbb{R}^+ for all $j = 1, \dots, p$.

Case 2: $\sharp C'_l = 2$.

Say $C'_l = \{\eta_l^{i_1}, \eta_l^{i_2}\}$ and, without loss of generality:

$$\begin{aligned} \eta_{l+1}^{i_1} &= \tau_1 + M_l, \\ \eta_{l+1}^{i_2} &= \tau_+^i - z^i + M_l, \\ \eta_{k+1}^{i'_1} &= \tau_+^j - z^j + M_l + w, \text{ and} \\ \eta_{k+1}^{i'_2} &= \tau_1 + M_l + w, \end{aligned}$$

for some $1 \neq i \neq j \neq 1$. Suppose that i and j are in phase and, say, $0 < i < j$. There is then q so that $i + q = m$ and $j + q = m + sp$, which we may take to be $m + p$. Now, from Property 1, $\tau_-^i - z^i + M_l \in C_l$ and $\tau_-^j - z^j + M_l + w \in C_k = C_l + w$. Thus $\tau_-^j - z^j + M_l \in C_l$. Say $\tau_-^i - z^i + M_l \in S_a$ and $\tau_-^j - z^j + M_l \in S_b$. Then $\tau_-^{m+1} - z^{m+1} + \beta^{q+1} M_l \in \Psi_\beta^{q+1}(S_a) \cap \Psi_\beta^{q+1}(S_b) = S_a \cap S_b$, in contradiction to the disjointness of S_a and S_b .

Thus, i and j are not in phase (so $p > 1$). It follows as in Case 1 that $\tau_+^i - z^i \sim_s \tau_+^j - z^j$ densely on $[0, \epsilon)$ for some $\epsilon > 0$ and then that $T_{i'} \sim_s T_{j'}$ densely on \mathbb{R}^+ , with $i' \neq j' \in \{1, \dots, p\}$ such that $i' + m$ is in phase with i and $j' + m$ is in phase with j . \square

Tilings T, T' are *asymptotic on* \mathbb{R}^+ provided $\lim_{t \rightarrow \infty} d(T - t, T' - t) = 0$. It follows from the Meyer property, which all Pisot substitution tilings enjoy, that $T, T' \in \Omega_{\psi_\beta}$ are asymptotic on \mathbb{R}^+ if and only if there is t_0 so that $B_0[T - t] = B_0[T' - t]$ for all $t > t_0$ (see [BO]). For a discussion of the fundamental role asymptotic tilings play in the classification of one-dimensional substitution tilings, and an algorithm for finding them, see [BD1].

The tilings T_j^0 are all pairwise asymptotic on \mathbb{R}^+ . At least for some β , Ω_{ψ_β} has additional asymptotic pairs.

Example 3. Consider $\beta = (3 + \sqrt{5})/2$. Then $\kappa(1) = 2\bar{1}$ and the substitution is

$$1 \rightarrow 121; \quad 2 \rightarrow 21.$$

The tilings T_1^0 and T_1 are asymptotic on \mathbb{R}^+ .

Example 4. Consider β with $\kappa(1) = 220\bar{1}$ and substitution

$$1 \rightarrow 12; \quad 2 \rightarrow 34; \quad 3 \rightarrow 2341; \quad 4 \rightarrow 23.$$

The tilings corresponding to the T_β -periodic points with itineraries $\overline{21}$ and $\overline{20}$ are asymptotic on \mathbb{R}^+ .

For an arbitrary substitution ϕ with prototiles ρ_1, \dots, ρ_d , the rose \mathcal{R}_ϕ is the wedge of circles obtained by identifying the vertices in the disjoint union of the prototiles: $\mathcal{R}_\phi = \cup_{i=1}^d \rho_i / \sim$. Let $g_\phi : \Omega_\phi \rightarrow \mathcal{R}_\phi$ be given by

$$(4.1) \quad g_\phi(T) = [(x, i)]$$

provided $x \in \text{spt}(\rho_i)$ and $\rho_i - x \in T$, and let

$$(4.2) \quad f_\phi : \mathcal{R}_\phi \rightarrow \mathcal{R}_\phi$$

be the continuous map induced by Φ ; that is, $f_\phi \circ g_\phi = g_\phi \circ \Phi$. For β substitutions, it will be convenient to also have the (discontinuous) map

$$(4.3) \quad g_I : \Omega_{\psi_\beta} \rightarrow I = [0, 1]$$

given by $g_I(T) = x$ provided $\min(\tau_j) \leq x < \max(\tau_j)$ and $\tau_j - x \in T$. Note that $T_\beta \circ g_I = g_I \circ \Psi_\beta$ and $g_{\psi_\beta} = q \circ g_I$, where $q : I \rightarrow \mathcal{R}_{\psi_\beta} \simeq I / \{0 = s_0 < s_1 < \dots < s_{m+p} = 1\}$ is the natural quotient map and $\{s_i : i = 0, \dots, s_{m+p}\} = \{0, 1\} \cup \mathcal{O}_{T_\beta}(1)$.

The following somewhat technical result is crucial to our analysis.

Proposition 5. *If $T \neq T' \in \Omega_{\psi_\beta}$ are asymptotic on \mathbb{R}^+ and are both Ψ_β -periodic of (least) period p , then either $\{T, T'\} \subset \{T_j^0 : j = 1, \dots, p\}$ or T and T' are in distinct Ψ_β -orbits.*

Proof. Let's suppose that $T \neq T' \in \Omega_{\psi_\beta}$ are asymptotic on \mathbb{R}^+ and Ψ_β -periodic of period p . Let $t_0 = t_0(T, T') := \inf\{t' : B_0[T - t] = B_0[T' - t] \text{ for all } t > t'\}$. Then there are $k \neq k'$ so that $B_0[T - t_0] = \{\tau_-^k - z^k + t_0, \tau_1 + t_0\}$ and $B_0[T' - t_0] = \{\tau_-^{k'} - z^{k'} + t_0, \tau_1 + t_0\}$. (The occurrence of the τ_1 in both patches follows from Property 3.) If $t_0(T, T') = 0$, then $\{T, T'\} \subset \{T_j^0 : j = 1, \dots, p\}$. We will assume from now on that $t_0(T, T') > 0$. Then k, k' must be *in phase*; that is, $p | (k - k')$, for otherwise, $t_0(T, T') = t_0(\Psi_\beta^p(T), \Psi_\beta^p(T')) = \beta^p t_0(T, T') > t_0(T, T')$. Since $t_{\psi_\beta}(r) = t_{\psi_\beta}(s)$ for $r \neq s$ only when $\max(\tau_r), \max(\tau_s) \in \{z^m, z^{m+p}\}$, $t_0(n) := t_0(\Psi_\beta^n(T), \Psi_\beta^n(T'))$ will increase with n until, for some n_0 , $\tau_-^m - z^m + t_0(n_0), \tau_-^{m+p} - z^{m+p} + t_0(n_0) \in B_0[\Psi_\beta^{n_0}(T) - t_0(n_0)] \cup B_0[\Psi_\beta^{n_0}(T') - t_0(n_0)]$. Let us examine the effect of applying Ψ_β to the patches in $\Psi_\beta^{n_0}(T)$ and $\Psi_\beta^{n_0}(T')$ that lie over $[0, t_0(n_0)]$.

First let's suppose that $z^{m+p} < z^m$ and, without loss, that $\tau_-^{m+p} - z^{m+p} + t(n_0) \in \Psi_\beta^{n_0}(T)$ and $\tau_-^m - z^m + t_0(n_0) \in \Psi_\beta^{n_0}(T')$. Let k, k' be such that $\tau_k = \tau_-^{m+p}$ and $\tau_{k'} = \tau_-^m$. From Property 1 we have that the patches $P := \{\tau_1, \dots, \tau_k\} - z^{m+p} + t_0(n_0)$ and $P' := \{\tau_1, \dots, \tau_{k'}\} - z^m + t_0(n_0)$ are contained in $\Psi_\beta^{n_0}(T)$ and $\Psi_\beta^{n_0}(T')$, respectively. Since $T_\beta(z^{m+p}) = T_\beta(z^m)$, we have that

$T_\beta(t + z^m - z^{m+p}) = T_\beta(t)$ for $0 \leq t \leq z^{m+p}$ and it follows that $\Psi_\beta(P) \subset \Psi_\beta(P')$. Let $\tau_-^l - z^l - z^{m+p} + t_0(n_0)$ be the tile in $\Psi_\beta^{n_0}(T)$ that immediately precedes P . We have that $t_0(n_0 + 1) = \beta t_0(n_0) - \beta z^{m+p}$ and $\tau_-^{l+1}, \tau_-^1 \in B_0[\Psi_\beta^{n_0+1}(T) - t_0(n_0 + 1)] \cup B_0[\Psi_\beta^{n_0+1}(T') - t_0(n_0 + 1)]$. Now $t_0(n)$ must increase for $n = n_0 + 1, \dots, n_0 + m - 1$ and this means that $m \leq p$. Thus, since $l + 1$ and 1 must be in phase, $l = p$.

An identical analysis with the same conclusion applies to the case $z^m < z^{m+p}$. To ease notation, we may then (replacing T and T' by $\Psi_\beta^{n_0+1}(T)$ and $\Psi_\beta^{n_0+1}(T')$) assume that $B_0[T - t_0(0)] = \{\tau_-^{p+1} - z^{p+1} + t_0(0), \tau_1 + t_0(0)\}$ and $B_0[T' - t_0(0)] = \{\tau_-^1 - 1 + t_0(0), \tau_1 + t_0(0)\}$. We see now that $B_0[\Psi_\beta^n(T) - t_0(n)] = \{\tau_-^{p+1} - z^{p+1} + t_0(n), \tau_1 + t_0(n)\}$ and $B_0[\Psi_\beta^n(T') - t_0(n)] = \{\tau_-^1 - 1 + t_0(n), \tau_1 + t_0(n)\}$ if and only if $n = km$ for some k . Hence $m|p$.

Let us next rule out $z^m < z^{m+p}$. Suppose that this is the case. The patch $P' := \{\tau_1, \tau_2, \dots, \tau_-^m\} - z^m + t_0(m - 1) \subset \Psi_\beta^{m-1}(T')$ is immediately preceded (in $\Psi_\beta^{m-1}(T')$) by a translate of some tile τ_-^l . Then $B_0[\Psi_\beta^m(T') - t_0(m)] = \{\tau_-^{l+1} - z^{l+1} + t_0(m), \tau_1 + t_0(m)\}$ while $B_0[\Psi_\beta^m(T) - t_0(m)] = \{\tau_-^1 - 1 + t_0(m), \tau_1 + t_0(m)\}$. Since $z^m < z^{m+p}$, m must be greater than 1 and then $z^m < \beta^{m-1}z^1$. It follows that the tile $\tau_1 - z^m + t_0(m - 1)$ in P' can't be the first tile in a patch $\Psi_\beta^{m-1}(\tau_1 + \beta^{1-m}(t_0(m - 1) - z^m))$ with $\tau_1 + \beta^{1-m}(t_0(m - 1) - z^m) \in T'$. Thus it must be the case that there is $k, 1 \leq k < m - 1$, and $\tau \in \Psi_\beta^k(T')$ with $\Psi_\beta(\tau) = \{\dots, \tau_-^1 - z^1 + t', \tau_1 + t', \dots\}$ and $\Psi_\beta^{m-2-k}(\tau_1 + t') = \{\tau_1 - z^m + t_0(m - 1), \dots\}$. But then $l = m - 1 - k$ so that $1 < l + 1 < m$ and then 1 and $l + 1$ are definitely out of phase.

Thus $z^{m+p} < z^m$.

For each $n \in \mathbb{N}$ we have:

- (1) $B_0[\Psi_\beta^{nm}(T) - t_0(nm)] = \{\tau_-^{p+1} - z^{p+1} + t_0(nm), \tau_1 + t_0(nm)\}$,
- (2) $B_0[\Psi_\beta^{nm}(T') - t_0(nm)] = \{\tau_-^1 - z^1 + t_0(nm), \tau_1 + t_0(nm)\}$,
- (3) $\{\tau_-^p - z^p, \tau_1, \tau_2, \dots, \tau_-^{m+p}\} - z^{m+p} + t_0(mn - 1) \subset \Psi_\beta^{mn-1}(T)$, and
- (4) $\{\tau_1, \tau_2, \dots, \tau_-^m\} - z^m + t_0(mn - 1) \subset \Psi_\beta^{mn-1}(T')$.

Claim 1: $t_0(nm) < 1$ for all n .

To see that this must be true, suppose that $b = t_0(nm) - 1 > 0$ for some n . Then the patch $P_n = \{\tau_1, \tau_2, \dots, \tau_-^1\} - z^1 + t_0(nm)$ in $\Psi_\beta^{nm}(T')$ has the property that $\Psi_\beta^m(P)$ contains the patch P_{n+1} in $\Psi_\beta^{(n+1)m}(T')$, so that $t_0((n + 1)m) \geq \beta^m b + 1$. That is, $t_0((n + 1)m) - 1 \geq \beta^m b$. Inductively, $t_0((n + k)m) \geq \beta^k b$ for $k \in \mathbb{N}$. But t_0 is periodic. If $t_0(nm) = 1$ for some n then $\tau_1 \in \Psi_\beta^{nm}(T')$ and $\tau_1 \in \Psi_\beta^{km}(T') = T'$, so $T' \neq T_j$ for any j . We have established Claim 1.

Let $y := g_I(T')$ (recall definition 4.3) and let $r := \lfloor \beta z^{m+p} \rfloor$. In other words, $y = 1 - t_0(0) = z^1 - t_0(0)$ and r is the number of discontinuities of T_β on $[0, z^{m+p}]$.

Claim 2: $\kappa(y) = \overline{a_1 a_2 \dots a_{m-1} (a_m - r - 1)}$.

To prove the claim, we first suppose that there is $j < m$ with $(\kappa(y))_j < a_j$. Then the patch $\{\tau_1, \tau_2, \dots, \tau_-^{j+1}\} - z^{j+1} + t_0(j)$ in $\Psi_\beta^j(T')$ has support contained in $[0, \infty)$. The patch $\{\tau_1, \tau_2, \dots, \tau_-^m\} - z^m + t_0(m - 1)$ of $\Psi_\beta^{m-1}(T')$ then also has support in $[0, \infty)$ and hence so does the patch $\{\tau_1, \tau_2, \dots, \tau_-^1\} - z^1 + t_0(m)$. This means that $t_0(m) \geq 1$, which has been ruled out by Claim 1. Thus $(\kappa(y))_j = a_j$ for $j = 1, \dots, m - 1$. It follows that $t_0(j) = z^{j+1} - T_\beta^j(y)$ for $j = 0, \dots, m - 2$

If $|t_0(m-1)-t| < z^{m+p}$, then $B_0[\Psi_\beta^m(T)-\beta t] = B_0[\Psi_\beta^m(T')-\beta t]$. Consequently, $t_0(m-1) > z^{m+p}$, which means that $T_\beta^{m-1}(y) < z^m - z^{m+p}$. For $t \in [0, z^{m+p}]$, $T_\beta(t) = T_\beta(t + z^m - z^{m+p})$. Thus T_β has r discontinuities on the interval $(z^m - z^{m+p}, z^m]$; it's also discontinuous at $z^m - z^{m+p}$. Thus T_β has at least $r+1$ discontinuities between $T_\beta^{m-1}(y)$ and z^m and we have that $(\kappa(y))_m \leq a_m - r - 1$. Suppose that $(\kappa(y))_m < a_m - r - 2$. Then $t_0(m-1) > z^m - z^{m+p} - 1/\beta$ and $t_0(m) \geq 1$, contrary to Claim 1. Thus, $(\kappa(y))_j$ is as claimed for $j = 1, \dots, m$. Repeating the argument periodically gives $\kappa(y) = \overline{a_1 a_2 \dots a_{m-1} (a_m - r - 1)}$, as claimed.

From Claim 2, y is T_β -periodic of period m . Thus T' is Ψ_β -periodic of period m , so $m = p$ and we have that $\kappa(y) = \overline{a_1 a_2 \dots a_{p-1} (a_p - r - 1)}$. Let's now determine itinerary $\kappa(x)$ of $x := g_I(T)$.

Consider the patch $\{\tau_-^p - z^p, \tau_1\} - z^{2p} + t_0(p-1) \subset \Psi_\beta^{p-1}(T)$ (see item (3), with $m = p, n = 1$) and note that $\tau_1 - z^{2p} + t_0(p)$ has support contained in \mathbb{R}^+ . This can only come from a patch $\{\tau_-^{p-1} - z^{p-1}, \tau_1\} - \beta^{-1}(z^{2p} - t_0(p-1))$ in $\Psi_\beta^{p-2}(T)$, which can only have come from a patch $\{\tau_-^{p-2} - z^{p-2}, \tau_1\} - \beta^{-2}(z^{2p} - t_0(p-1))$ in $\Psi_\beta^{p-3}(T), \dots$. We see that the patch $\{\tau_-^1 - 1, \tau_1, \tau_2, \dots, \tau_-^{p+1}\} - z^{p+1} + t_0(0)$ is contained in T and the support of $\tau_1 - z^{p+1} + t_0(0)$ is contained in \mathbb{R}^+ . The origin is then located in the interior of the support of the patch $\{\tau_1, \tau_2, \dots, \tau_-^1\} - 1 - z^{p+1} + t_0(0) \subset T$ (see Claim 1) at relative position x . That is, $x = 1 + z^{p+1} - t_0(0)$. Since, under application of Ψ_β^{p-1} , the tile $\tau_-^1 - 1 - z^{p+1} + t_0(0)$ produces the patch $\{\dots, \tau_-^p - z^p - z^{2p} + t_0(p-1)\}$ in $\Psi_\beta^{p-1}(T)$ and the support of the latter patch is contained in $(-\infty, t_0(p-1)]$, we have $\beta^j(\max(\tau_-^1 - 1 - z^{p+1} + t_0(0))) = \beta^j(t_0(0) - z^{p+1}) \leq t_0(j)$ for $j = 0, \dots, p-1$. If there were a discontinuity of T_β between $T_\beta^j(x)$ and z^{j+1} for some $j \in \{0, \dots, p-2\}$, then $t_0(j+1)$ would be at least 1, which is not allowed by Claim 1. Thus $(\kappa(x))_j = a_j$ for $j = 1, \dots, p-1$.

One can show that $\kappa(x) = \overline{a_1 a_2 \dots a_{p-1} (a_p - 1)}$ (as in Example 4) but at this point we have all we need to know that the tilings $T \neq T'$ are not in the same Ψ_β -orbit. Indeed, since $\Psi_\beta \circ g_I = g_I \circ T_\beta$, $\Psi_\beta^k(T) = T'$ would imply $\sigma^k(\kappa(g_I(T))) = \kappa(g_I(T'))$, which would mean that $\sum_{j=1}^p (\kappa(g_I(T)))_j = \sum_{j=1}^p (\kappa(g_I(T')))_j$. The latter equality can only hold if $g_I(T) = g_I(T')$ (since we have shown that $(\kappa(g_I(T)))_j = (\kappa(g_I(T')))_j$ for $j = 1, \dots, p-1$), but then $T = T'$. \square

Corollary 6. *If $i \neq j$, then T_i and T_j are not asymptotic on \mathbb{R}^+ .*

Lemma 7. *If $m > 0$ then $T_i \sim_s T_j \sim_s T_k^0$ densely on \mathbb{R}^+ for all $i, j, k \in \{1, \dots, p\}$.*

Proof. If $p = 1$ this is the second statement of Lemma 2. If $p > 1$, let $i \neq j$ be so that $T_i \sim_s T_j$ densely on \mathbb{R}^+ (from the first statement of Lemma 2). From Corollary 6 we know that T_i and T_j are not asymptotic on \mathbb{R}^+ . Thus there are patches $\{\tau, \tau'\} \subset T_i$ and $\{\tau, \tau''\} \subset T_j$ with support contained in \mathbb{R}^+ , $\tau' \neq \tau''$, and $\min(\tau') = \min(\tau'') = \max(\tau)$. From Property 2, one of τ', τ'' is of type 1: say $\tau' = \tau_1 + t$ and $\tau'' = \tau_r - \min(\tau_r) + t$, with $r \in \{2, \dots, m+p\}$. Let $l \in \{m+1, \dots, m+p\}$ be in phase with r (that is, $p \mid (l-r)$). Then $\Psi_\beta^{qp}(\tau'') = \{\tau_l - \min(\tau_l) + \beta^{qp}t, \dots\}$ for some q . Let $\epsilon = \min\{\text{length}(\tau_1), \text{length}(\tau_r), \beta^{-qp} \cdot \text{length}(\tau_l)\}$. Then $\tau_1 \sim_s \tau_r - \min(\tau_r) \sim_s \tau_l - \min(\tau_l)$ densely on $[0, \epsilon)$. It follows that $T_l \sim_s T_k^0$ densely on \mathbb{R}^+ for all $k \in \{1, \dots, p\}$. Thus, with subscripts taken mod (p) , $T_{l+n} = \Psi_\beta^n(T_l) \sim_s \Psi_\beta^n(T_k^0) = T_{k+n}^0$ densely on \mathbb{R}^+ for all $n \in \mathbb{N}$. \square

The *stable equivalence relation*, \approx_s , is defined for a substitution tiling space Ω_ϕ by $T \approx_s T'$ if and only if $T \sim_s T'$ densely on \mathbb{R} and $T \sim_{srp} T'$.

Corollary 8. *If $m > 0$ then $T_j \approx_s T_j^0$ for each $j \in \{1, \dots, p\}$.*

The following is Theorem 5 of [B3].

Theorem 9. *If ϕ is a primitive, non-periodic Pisot substitution, then $(\Omega_{\psi_\beta}, \mathbb{R})$ has pure discrete spectrum if and only if stable equivalence equals strong regional proximality on Ω_{ψ_β} .*

Given a morphism γ from letters of an alphabet \mathcal{A} to nonempty words on an alphabet \mathcal{A}' , let $i_\gamma, t_\gamma : \mathcal{A} \rightarrow \mathcal{A}'$ be the initial and terminal letter maps: If $\gamma(i) = a \cdots z$ then $i_\gamma(i) = a$ and $t_\gamma(i) = z$.

The stable equivalence relation is pushed down to a relation R on the rose \mathcal{R}_ϕ by xRy if and only if there are $T_1, \dots, T_n, T'_1, \dots, T'_n \in \Omega_\phi$ with $T_i \approx_s T'_i$ for $i = 1, \dots, n$, $g_\phi(T_{i+1}) = g_\phi(T'_i)$ for $i = 1, \dots, n-1$, $g_\phi(T_1) = x$, and $g_\phi(T'_n) = y$. The following lemma is proved in [B1] under the slightly stronger assumption that either t_ϕ or i_ϕ is eventually constant (see Lemmas 3.5 and 3.6 there), but the proof goes through virtually without change under the condition we assume here of dense terminal or dense initial stable relation.

Lemma 10. *Suppose that ϕ is a nonperiodic Pisot substitution with the properties: $(\Omega_\phi, \mathbb{R})$ does not have pure discrete spectrum and there is $\epsilon > 0$ so that either $\tau - \min(\tau) \sim_s \tau' - \min(\tau')$ densely on $[0, \epsilon)$ for all tiles τ, τ' , or $\tau - \max(\tau) \sim_s \tau' - \max(\tau')$ densely on $(-\epsilon, 0]$ for all tiles τ, τ' . Then R is a closed equivalence relation on \mathcal{R}_ϕ with boundedly finite equivalence classes.*

Recall (definition 4.2) that $f_\phi : \mathcal{R}_\phi \rightarrow \mathcal{R}_\phi$ is the map induced by $\Phi : \Omega_\phi \rightarrow \Omega_\phi$ (so that $f_\phi \circ g_\phi = g_\phi \circ \Phi$). Then $xRy \implies f_\phi(x)Rf_\phi(y)$ and there is an induced map $\tilde{f}_\phi : \mathcal{R}_\phi/R \rightarrow \mathcal{R}_\phi/R$. Under the assumptions of Lemma 10, \mathcal{R}_ϕ/R is a wedge of circles, \tilde{f}_ϕ defines a substitution $\tilde{\phi}$, and we may identify \mathcal{R}_ϕ/R with $\mathcal{R}_{\tilde{\phi}}$. If \mathcal{A} and $\tilde{\mathcal{A}}$ are the alphabets for ϕ and $\tilde{\phi}$ then the quotient map from \mathcal{R}_ϕ to $\mathcal{R}_{\tilde{\phi}}$ induces a morphism $\alpha : \mathcal{A} \rightarrow \tilde{\mathcal{A}}^*$ so that $\alpha \circ \phi = \tilde{\phi} \circ \alpha$.

The following is distilled from [B1] (see, in particular, Theorem 3.9 there).

Theorem 11. *Suppose that ϕ is a nonperiodic Pisot substitution with the properties: $(\Omega_\phi, \mathbb{R})$ does not have pure discrete spectrum and there is $\epsilon > 0$ so that either $\tau - \min(\tau) \sim_s \tau' - \min(\tau')$ densely on $[0, \epsilon)$ for all tiles τ, τ' ; or $\tau - \max(\tau) \sim_s \tau' - \max(\tau')$ densely on $(-\epsilon, 0]$ for all tiles τ, τ' . Then $\tilde{\phi}$ is a nonperiodic Pisot substitution and $(\Omega_{\tilde{\phi}}, \mathbb{R})$ does not have pure discrete spectrum.*

To apply Theorem 11 to β -substitutions we will require the following lemma.

Lemma 12. *If $m > 0$, there is $\epsilon > 0$ so that $\tau - \min(\tau) \sim_s \tau' - \min(\tau')$ densely on $[0, \epsilon)$ for all tiles τ, τ' for ψ_β .*

Proof. For each $i > 1$ there is $j \in \{1, \dots, p\}$ so that z^i is in phase with z^{m+j} . This means that $\Psi_\beta^{m-i+1}(\tau_+^i) = \{\tau_+^{m+1}, \dots\}$ and $\Psi_\beta^{m-i+1}(\tau_+^{m+j}) = \{\tau_+^{m+1}, \dots\}$. Hence $\tau^i \sim_s \tau^{m+j}$ densely on $[0, \epsilon_i)$, with $\epsilon_i := \beta^{i-m-1}(\text{length}(\tau^{m+1}))$. Since $T_i \sim_s T_j \sim_s T_k^0$ densely on \mathbb{R}^+ (Lemma 7) we have that $\tau_1 \sim_s \tau_+^l$ densely on $[0, \epsilon)$, with $\epsilon := \min\{\beta^{-m-1}(\text{length}(\tau_+^{m+1})), \text{length}(\tau_1)\}$, for all $l \in \{2, \dots, m+p\}$. \square

Lemma 13. *Suppose that $(\Omega_{\psi_\beta}, \mathbb{R})$ does not have pure discrete spectrum and $m > 0$. Then $i_{\tilde{\psi}_\beta}$ is eventually constant and $t_{\tilde{\psi}_\beta}$ is injective.*

Proof. Pick $i \in \{1, \dots, p\}$ and let $P \subset T_i$ and $P' \subset T_i^0$ be patches, both supported on $[0, t_0]$, with $P \cap P' = \emptyset$ and so that $\tau_1 + t_0 \in B_0[T_i - t_0] \cap B_0[T_i^0 - t_0]$ (that is, P, P' are the initial patches of T_i, T_i^0 preceding their first agreement). Then $\Psi_\beta^p(P) = P \cup \{\tau_1 + t_0\} \cup Q$ and $\Psi_\beta^p(P') = P' \cup \{\tau_1 + t_0\} \cup Q'$. Let $k > 0$ be minimal with the property that $\psi_\beta^k(P) \cap \psi_\beta^k(P') \neq \emptyset$. We may as well assume

(after replacing T_i and T_i^0 by $\Psi_\beta^{k-1}(T_i)$ and $\Psi_\beta^{k-1}(T_i^0)$, and P, P' by the initial disjoint patches of $\Psi_\beta^{k-1}(T_i), \Psi_\beta^{k-1}(T_i^0)$) that $k = 1$. Then $\Psi_\beta(P) = P_1 \cup \{\tau_1 + t\} \cup P_2$ and $\Psi_\beta(P') = P'_1 \cup \{\tau_1 + t\} \cup P'_2$ for some $t > 0$, with $P_1 \cap P'_1 = \emptyset$ and $\text{spt}(P_1) = [0, t] = \text{spt}(P'_1)$. Consider the (disjoint) patches $B_0[T_i - t/\beta] \subset P$ and $B_0[T_i^0 - t/\beta] \subset P'$. Let us first observe that these patches can't both be doubletons. Indeed, if $B_0[T_i - t/\beta] = \{\rho_1 < \rho_2\}$ and $B_0[T_i^0 - t/\beta] = \{\rho'_1 < \rho'_2\}$, then $\rho_2 = \tau_1 + t/\beta = \rho'_2$ (since τ_1 is the only prototile whose image under Ψ_β begins with a tile of type 1), contradicting the disjointness of P and P' . If $B_0[T_i - t/\beta]$ and $B_0[T_i^0 - t/\beta]$ are both singletons, then these must be of the form $\{\tau\}, \{\tau'\}$ with τ a translate of some τ_l and τ' a translate of some $\tau_{l'}$ with l, l' such that $k/\beta \in \text{int}(\text{spt}(\tau_l))$ and $k'/\beta \in \text{int}(\text{spt}(\tau_{l'}))$ for some $k, k' \in \{1, \dots, \lfloor \beta \rfloor\}$. But then $\tau_{m+p} + t - 1 \in \Psi_\beta(\tau) \cap \Psi_\beta(\tau')$ is in both P_1 and P'_1 , contradicting the disjointness of these patches. Thus, exactly one of $B_0[T_i - t/\beta]$ and $B_0[T_i^0 - t/\beta]$ is a doubleton and one of the patches P_1, P'_1 ends in $\tau_{m+p} + t - 1$ and the other ends in $\tau_k + t - \max(\tau_k)$ for some $k \in \{1, \dots, m+p-1\}$.

Since $\Psi_\beta(T_i) \approx_s \Psi_\beta(T_i^0)$, we see from the definition of the relation R that $q(1-s)Rq(\max(\tau_k) - s)$ in the rose \mathcal{R}_β for all positive s less than the smaller of the lengths of τ_k and τ_{m+p} . This means that $t_\alpha(k) = t_\alpha(m+p)$. Let $\max(\tau_k) = z^{n+1}$. Then $t_{\psi_\beta^n}(m+p) = k$ and it follows from $\alpha \circ \psi_\beta = \tilde{\psi}_\beta \circ \alpha$ that $t_{\tilde{\psi}_\beta^n}(t_\alpha(m+p)) = t_\alpha(t_{\psi_\beta^n}(m+p)) = t_\alpha(k) = t_\alpha(m+p)$. Now let a be any letter in the alphabet for $\tilde{\psi}_\beta$ and let $l \in \{1, \dots, m+p\}$ be so that $t_\alpha(l) = a$. Let $r \in \mathbb{N}$ be so that $t_{\psi_\beta^r}(m+p) = l$. Then

$$\begin{aligned}
t_{\tilde{\psi}_\beta^n}(a) &= t_{\tilde{\psi}_\beta^n}(t_\alpha(l)) \\
&= t_{\tilde{\psi}_\beta^n}(t_\alpha(t_{\psi_\beta^r}(m+p))) \\
&= t_{\tilde{\psi}_\beta^n}(t_\alpha(t_{\psi_\beta^n}(m+p))) \\
&= t_{\tilde{\psi}_\beta^r}(t_\alpha(m+p)) \\
&= t_\alpha(t_{\psi_\beta^r}(m+p)) \\
&= t_\alpha(l) \\
&= a.
\end{aligned}$$

Hence $t_{\tilde{\psi}_\beta}$ is injective.

Pick $j \in \{1, \dots, p\}$ (so $T_j \approx_s T_j^0$ by Corollary 8) and let $k \in \{2, \dots, m+p\}$ be so that $\min(\tau_k) = z^{m+j}$ (thus τ_k is the initial tile of T_j). Then $i_\alpha(k) = i_\alpha(1)$. Now, for any $l \in \{2, \dots, m+p\}$ there is $n \in \mathbb{N}$ so that $i_{\psi_\beta^n}(l) = k$, and then $i_{\tilde{\psi}_\beta^n}(i_\alpha(l)) = i_\alpha(i_{\psi_\beta^n}(l)) = i_\alpha(k) = i_\alpha(1)$. Since $i_{\psi_\beta}(1) = 1$, $i_{\tilde{\psi}_\beta}(i_\alpha(1)) = i_\alpha(1)$. That $i_{\tilde{\psi}_\beta}$ is eventually constant (with value $i_\alpha(1)$) follows from surjectivity of i_α . □

The following is Theorem 3.12 of [B1].

Theorem 14. *Suppose that ϕ is a nonperiodic Pisot substitution with the properties: i_ϕ is eventually constant and t_ϕ is injective. Then $(\Omega_\phi, \mathbb{R})$ has pure discrete spectrum.*

Theorem 15. *If β is Pisot, then $(\Omega_{\psi_\beta}, \mathbb{R})$ has pure discrete spectrum.*

Proof. This is clearly true if $\beta = n \in \mathbb{N}$. Suppose that β is Pisot, and not an integer. If $m = 0$ it is easy to check that i_{ψ_β} is eventually constant and t_{ψ_β} is injective. Hence $(\Omega_{\psi_\beta}, \mathbb{R})$ has pure discrete spectrum by Theorem 14. If $m > 0$ and $(\Omega_{\psi_\beta}, \mathbb{R})$ does not have pure discrete spectrum, then the substitution $\tilde{\psi}_\beta$ is non-periodic Pisot and $(\Omega_{\tilde{\psi}_\beta}, \mathbb{R})$ does not have pure discrete spectrum (Lemma 12 to get $\tilde{\psi}_\beta$ and Theorem 11). But $(\Omega_{\tilde{\psi}_\beta}, \mathbb{R})$ must have pure discrete spectrum by Lemma 13 and Theorem 14. \square

Remark 16. *If the degree d of the Pisot number β equals $m + p$ (so that the β -substitution is irreducible), then it follows from Theorem 15 and [CS] (see, also, [Sa]) that the substitutive system (not to be confused with the β -shift) also has pure discrete spectrum. There are, however, examples of Pisot β for which the substitutive system does not have pure discrete spectrum ([EI]). It follows from Theorem 15 and Proposition 8.1 of [BBK] that, for Pisot units, the substitutive system associated with ψ_β has, as an a.e. one-to-one factor, an isometric exchange of domains (the Rauzy pieces) induced as a first return of a minimal translation on the $(d - 1)$ -torus.*

Corollary 17. *All Pisot numbers are weakly finitary.*

Corollary 18. *If β is Pisot, M is the companion matrix for the minimal polynomial of β , and \bar{y} is a fundamental homoclinic point for the solenoidal automorphism \hat{F}_M , then the arithmetical coding $h_{\bar{y}} : X_\beta \rightarrow \hat{\mathbb{T}}_\beta^d$ is a.e. one-to-one.*

We prove Corollaries 17 and 18 in the next section.

5. THE CONNECTION BETWEEN PURE DISCRETE SPECTRUM, ARITHMETICAL CODING, AND PROPERTY (W).

For this section, fix a Pisot number β of algebraic degree d . The *substitution matrix* A associated with ψ_β is the $(m + p) \times (m + p)$ matrix with ij -th entry a_{ij} given by the number of occurrences of the letter j in the word $\psi_\beta(i)$. Then β is a simple eigenvalue of A and the characteristic polynomial of A factors over \mathbb{Z} as

$$p_A(x) = p_\beta(x)q(x)$$

with $p_\beta(x)$ the minimal polynomial of β and $q(x)$ relatively prime with $p_\beta(x)$. There are then $s_1(x), s_2(x) \in \mathbb{Q}[x]$ with $s_1(x)p_\beta(x) + s_2(x)q(x) = 1$ and an A -invariant splitting

$$\mathbb{R}^{m+p} = V \oplus W,$$

where $V := \ker(s_1(A)p_\beta(A))$ and $W := \ker(s_2(A)q(A))$ are rational (that is, spanned by rational vectors). The *Pisot subspace* V is d -dimensional and the linear transformation $L := A|_V$ has characteristic polynomial $p_\beta(x)$.

Let $V = E^s \oplus E^u$ be the splitting of V into $(d - 1)$ -dimensional stable and 1-dimensional unstable spaces, let $\pi_V : \mathbb{R}^{m+p} \rightarrow V$ be projection along W , and, for $v \in V$, write $v = v^s + v^u$ with $v^{s,u} \in E^{s,u}$. By π^s (resp., π^u) we will denote both projection of \mathbb{R}^{m+p} and V along $W \oplus E^u$ (resp., $W \oplus E^s$) and of V along E^u (resp., E^s) onto E^s (resp., E^u). The subgroup

$$\Gamma := \pi_V(\mathbb{Z}^{m+p})$$

of V is a d -dimensional lattice.

It is convenient to view tilings $T \in \Omega_{\psi_\beta}$ as maps of \mathbb{R} into the rose \mathcal{R}_{ψ_β} , rather than as collections of tiles: Given $T \in \Omega_{\psi_\beta}$, let $T(t) := g_{\psi_\beta}(T - t)$ (recall the definition 4.1). The \mathbb{R} -action on tilings-as-maps is then given by $(T - t)(s) := T(s + t)$; Ψ_β translates as $\Psi_\beta(T)(t) := f_{\psi_\beta}(T(t/\beta))$ (with

f_{ψ_β} as in 4.2); and the tiling metric induces the topology of uniform convergence on compact sets. We view the prototiles as parameterizations of the corresponding petals of the rose via $\tau_i(t) := T(t - \min(\tau_i))$ where $\tau_i \in T \in \Omega_{\psi_\beta}$.

For each $i \in \{1, \dots, m+p\}$, let $e_i \in \mathbb{Z}^{m+p}$ denote the standard unit vector and let $\sigma_i := \{te_i : t \in [0, 1]\} \subset \mathbb{R}^{m+p}$. Let $\tilde{\mathcal{R}}_{\psi_\beta} \subset \mathbb{R}^{m+p}$ be the grid

$$\tilde{\mathcal{R}}_{\psi_\beta} := \bigcup_{a \in \mathbb{Z}^{m+p}} \bigcup_{i=1, \dots, m+p} (a + \sigma_i).$$

The map $\tilde{\pi} : \tilde{\mathcal{R}}_{\psi_\beta} \rightarrow \mathcal{R}_{\psi_\beta}$ given by

$$\tilde{\pi}(a + te_i) := \tau_i(t)$$

is the *universal abelian cover* of \mathcal{R}_{ψ_β} .

We denote by $\tilde{f}_{\psi_\beta} : \tilde{\mathcal{R}}_{\psi_\beta} \rightarrow \tilde{\mathcal{R}}_{\psi_\beta}$ the unique lift of f_{ψ_β} with $\tilde{f}_{\psi_\beta}(0) = 0$ and, if $T \in \Omega_{\psi_\beta}$, $\tilde{T} : \mathbb{R} \rightarrow \tilde{\mathcal{R}}_{\psi_\beta}$ will denote any lift of $T : \mathbb{R} \rightarrow \mathcal{R}_{\psi_\beta}$. We then define $\tilde{\Psi}_\beta$ on such lifts by

$$\tilde{\Psi}_\beta(\tilde{T})(t) := \tilde{f}_{\psi_\beta} \circ \tilde{T}(t/\beta).$$

Note that since A is the ‘abelianization’ of f_{ψ_β} , $\tilde{f}_{\psi_\beta}(a) = Aa$ for $a \in \mathbb{Z}^{m+p}$.

By a *strand* γ corresponding to $T \in \Omega_{\psi_\beta}$ we will mean any

$$\gamma = x + \pi_V \circ \tilde{T},$$

with $x \in V$ and \tilde{T} a lift of T . For $T \in \Omega_{\psi_\beta}$, let

$$(5.1) \quad t_*(T) := \sup\{t \leq 0 : \tau_1 + t \in T\}.$$

So $T(t_*(T)) = *$ is the branch point of \mathcal{R}_{ψ_β} and $t_*(T)$ is the largest non-positive occurrence of the initial vertex of a tile of type 1 in T . For a strand γ corresponding to T we define its *initial vertex* to be

$$(5.2) \quad a(\gamma) := \gamma(t_*(T))$$

and the (abelian) *prefix* of T , $p(T)$, to be

$$(5.3) \quad p(T) := \gamma(t_*(T)) - \gamma(\beta t_*(\Psi_\beta^{-1}(T))).$$

Let us extend Ψ_β to strands by setting $\Psi_\beta(x + \pi_V \circ \tilde{T}) := Lx + \pi_V \circ \tilde{\Psi}_\beta(\tilde{T})$. Then, if γ is a strand corresponding to T , $\Psi_\beta(\gamma)$ is the unique strand corresponding to $\Psi_\beta(T)$ having initial vertex $a(\gamma') = p(\Psi_\beta(T)) + L\gamma(t_*(T))$. It is clear that Ψ_β is invertible on strands (even though, if $\det(L) \neq \pm 1$, $\tilde{\Psi}_\beta$ is not invertible), and

$$p(T) = a(\gamma) - La(\Psi_\beta^{-1}(\gamma))$$

for any strand γ corresponding to T . We extend the \mathbb{R} -action to lifts and strands by $(\tilde{T} - t)(s) := \tilde{T}(s + t)$ and $(\gamma - t)(s) := \gamma(s + t)$. One easily checks that $\tilde{\Psi}_\beta(\tilde{T} - t) = \tilde{\Psi}_\beta(\tilde{T}) - \beta t$ and $\Psi_\beta(\gamma - t) = \Psi_\beta(\gamma) - \beta t$.

Given a strand γ corresponding to $T \in \Omega_{\psi_\beta}$, consider the sequence $(a_i(\gamma))_{i \in \mathbb{Z}}$ of initial vertices of the strands in the Ψ_β -orbit of γ :

$$a_i(\gamma) := \Psi_\beta^i(\gamma)(t_*(\Psi_\beta^i(T))).$$

We see that $|a_i(\gamma) - La_{i-1}(\gamma)| = |p(\Psi_\beta^i(T))|$ is bounded. From the hyperbolicity of L it follows that $(a_i(\gamma))$ is *globally shadowed* by a unique point in V :

Lemma 19. *Given a strand γ , there is a unique point $G(\gamma) \in V$ with the property that $|L^i G(\gamma) - a_i(\gamma)|$, $i \in \mathbb{Z}$, is bounded. Furthermore, $a(\gamma) - G(\gamma)$ is uniformly bounded as a function of γ .*

Proof. We have $L^{-i}a_i^u(\gamma) = a_0^u(\gamma) + L^{-1}p^u(\Psi_\beta^0(T)) + \dots + L^{-i}p^u(\Psi_\beta^{i-1}(T))$ and $L^i a_{-i}^s(\gamma) = a_0^s(\gamma) - p^s(\Psi_\beta^{-1}(T)) - \dots - L^{i-1}p^s(\Psi_\beta^{-i}(T))$. Since the $p(\Psi_\beta^i(T))$ are bounded, the limits in the following formula for $G(\gamma)$ exist: $G(\gamma) = \lim_{i \rightarrow \infty} L^{-i}a_i^u(\gamma) + \lim_{i \rightarrow \infty} L^i a_{-i}^s(\gamma)$. \square

Note that if $\gamma' = x + \gamma$ is another strand corresponding to T , then $a_i(\gamma') = L^i x + a_i(\gamma)$, so $G(\gamma') = G(\gamma) + x$. Thus

$$\gamma_T := \gamma - G(\gamma)$$

is the unique strand corresponding to T with the property that $(a_i(\gamma_T))$ is bounded. We define $\pi : \Omega_{\psi_\beta} \rightarrow V/\Gamma$ by

$$(5.4) \quad \pi(T) := a_0(\gamma_T) + \Gamma.$$

Let $l = (l_1, \dots, l_{m+p})$ with $l_i := \text{length}(\tau_i)$ the length of the i -th prototile. Then l is a left eigenvector of A (for the eigenvalue β) and is orthogonal to E^s . Thus, setting ω to be the right positive eigenvector of A , normalized with $\langle l^t, \omega \rangle = 1$, we have $v^u = \pi^u(v) = \langle l^t, v \rangle \omega$ for all $v \in V$. It follows that if γ is any strand, then

$$(5.5) \quad \gamma^u(t) - \gamma^u(t') = (t - t')\omega$$

for all $t, t' \in \mathbb{R}$.

Lemma 20. $\pi(T - t) = \pi(T) - (t\omega + \Gamma)$ for all $T \in \Omega_{\psi_\beta}$ and $t \in \mathbb{R}$.

Proof. Since $\Psi_\beta^i(\gamma - t) = \Psi_\beta^i(\gamma) - \beta^i t$ for any strand γ , $i \in \mathbb{Z}$ and $t \in \mathbb{R}$, $|a_i(\gamma - t) - a_i(\gamma)|$ is bounded (for fixed γ, t) for $i \leq 0$. Now $|a_i(\gamma) - \Psi_\beta^i(\gamma)(0)|$ and $|a_i(\gamma - t) - \Psi_\beta^i(\gamma - t)(0)|$ are bounded and $(\Psi_\beta^i(\gamma) - \beta^i t)^u(0) - (\Psi_\beta^i(\gamma))^u(0) = \beta^i t\omega$. Thus $|a_i^u(\gamma - t) - a_i^u(\gamma) - \beta^i t\omega|$ is bounded for $i \geq 0$. It follows that $G(\gamma - t) = G(\gamma) - t\omega$. Note that $a_0(\gamma - t) \equiv a_0(\gamma) \pmod{\Gamma}$. We have $\pi(T - t) = a_0(\gamma_{T-t}) + \Gamma = a_0((\gamma_T - t) - G(\gamma_T - t)) + \Gamma = a_0(\gamma_T) - G(\gamma_T) - t\omega + \Gamma = \pi(T) - (t\omega + \Gamma)$ (using $G(\gamma_T) = 0$). \square

Lemma 21. $\pi : \Omega_{\psi_\beta} \rightarrow V/\Gamma$ is a continuous surjection.

Proof. Continuity is clear from the explicit formula for G given in the proof of Lemma 19. Surjectivity is a consequence of Lemma 20 and the irrationality of ω : If $cl\{t\omega : t \in \mathbb{R}\}$ were a proper sub-torus \mathbb{T} of dimension $k < d$, then there would be a rational k -dimensional subspace U of V , invariant under L . But then the degree of β would be at most k , and not d . \square

The previous two lemmas combine to say that π is factor map of $(\Omega_{\psi_\beta}, \mathbb{R})$ onto $(\mathbb{T}^d, \mathbb{R})$, with the \mathbb{R} -action on $\mathbb{T}^d \simeq V/\Gamma$ given by $(x + \Gamma) - t := x - t\omega + \Gamma$. It is easy to check that π also semiconjugates Ψ_β with $F_L : V/\Gamma \rightarrow V/\Gamma$ by $F_L(x + \Gamma) = Lx + \Gamma$. Thus π induces

$$(5.6) \quad \hat{\pi} : \Omega_{\psi_\beta} \rightarrow \hat{\mathbb{T}}^d := \varprojlim F_L,$$

which factors $(\Omega_{\psi_\beta}, \mathbb{R})$ onto $(\hat{\mathbb{T}}^d, \mathbb{R})$.

For $T, T' \in \Omega_{\psi_\beta}$, we say that T *globally shadows* T' , and write $T \sim_{gs} T'$, provided there are lifts \tilde{T}, \tilde{T}' of T, T' so that the strands $\gamma := \pi_V(\tilde{T}), \gamma' := \pi_V(\tilde{T}')$ satisfy: $|a_i(\gamma) - a_i(\gamma')|$ is bounded.

If β is a unit, then $\det(L) = \pm 1$, $\tilde{\Psi}_\beta$ is invertible on $\{\tilde{T} : T \in \Omega_{\psi_\beta}\}$, and the above definition can be rephrased as: $T \sim_{gs} T'$ provided there are lifts \tilde{T}, \tilde{T}' so that $|\pi_V \circ \tilde{\Psi}_\beta^i(\tilde{T})(t_{*,i}) - \pi_V \circ \tilde{\Psi}_\beta^i(\tilde{T}')(t'_{*,i})|$ is bounded, where $t_{*,i} := t_*(\Psi_\beta^i(T))$ and $t'_{*,i} := t_*(\Psi_\beta^i(T'))$. To cover the ‘nonunimodular’ case as well, we will need the following awkward (but straightforward) reformulation.

Lemma 22. *$T \sim_{gs} T'$ if and only if there are $B < \infty$ and lifts \tilde{T}, \tilde{T}' , so that for each $i \in \mathbb{N}$ there are $v_i \in \Gamma$ and lifts $\tilde{T}_i, \tilde{T}'_i$ of $\Psi_\beta^{-i}(T), \Psi_\beta^{-i}(T')$, such that $\tilde{\Psi}_\beta^i(\tilde{T}_i) = v_i + \tilde{T}$, $\tilde{\Psi}_\beta^i(\tilde{T}'_i) = v_i + \tilde{T}'$, and $|\pi_V \circ \tilde{\Psi}_\beta^k(\tilde{T}_i)(t_{*,k-i}) - \pi_V \circ \tilde{\Psi}_\beta^k(\tilde{T}'_i)(t'_{*,k-i})| \leq B$ for all $k \in \mathbb{N}$, where $t_{*,j} := t_*(\Psi_\beta^j(T))$ and $t'_{*,j} := t_*(\Psi_\beta^j(T'))$.*

Lemma 23. *$\hat{\pi}(T) = \hat{\pi}(T')$ if and only if $T \sim_{gs} T'$.*

Proof. Suppose that $\hat{\pi}(T) = \hat{\pi}(T')$ and fix $i \geq 0$. Let $\gamma_{\Psi_\beta^{-i}(T)} = x + \pi_V \circ \tilde{T}_i$ and $\gamma_{\Psi_\beta^{-i}(T')} = x' + \pi_V \circ \tilde{S}_i$ with \tilde{T}_i and \tilde{S}_i lifts of $\Psi_\beta^{-i}(T)$ and $\Psi_\beta^{-i}(T')$, resp. Then $\pi(\Psi_\beta^{-i}(T)) = \pi(\Psi_\beta^{-i}(T'))$ means that $x - x' \in \Gamma$. There is B so that $|a(\gamma_T)| < B/2$ for all T (Lemma 19), so $|(a_k(\gamma_{\Psi_\beta^{-i}(T)}) - x) - (a_k(\gamma_{\Psi_\beta^{-i}(T')}) - x)| = |(a_k(\gamma_{\Psi_\beta^{-i}(T)} - x) - (a_k(\gamma_{\Psi_\beta^{-i}(T')} - x))| < B$ for all $i, k \geq 0$. Let $y \in \mathbb{Z}^{m+p}$ be such that $\pi_V(y) = x - x'$ and set $\tilde{T}'_i := y + \tilde{S}_i$. Then, with $t_{*,j} := t_*(\Psi_\beta^j(T))$ and $t'_{*,j} := t_*(\Psi_\beta^j(T'))$, we have $|\pi_V \circ \tilde{\Psi}_\beta^k(\tilde{T}_i)(t_{*,k-i}) - \pi_V \circ \tilde{\Psi}_\beta^k(\tilde{T}'_i)(t'_{*,k-i})| = |(a_k(\gamma_{\Psi_\beta^{-i}(T)} - x) - (a_k(\gamma_{\Psi_\beta^{-i}(T')} - x))| < B$ for $k \in \mathbb{N}$. Letting $\tilde{T} = \tilde{T}_0$ and $\tilde{T}' = \tilde{T}'_0$ we have $\tilde{\Psi}_\beta^i(\tilde{T}_i) = \tilde{T} + v_i$ and $\tilde{\Psi}_\beta^i(\tilde{T}'_i) = \tilde{T}' + v_i$ for some $v_i \in \Gamma$; hence $T \sim_{gs} T'$.

Now suppose that $T \sim_{gs} T'$ and let $B > 0$, $v_i, \tilde{T}, \tilde{T}', \tilde{T}_i$ and \tilde{T}'_i be as in the reformulation of \sim_{gs} (Lemma 22). Let $\gamma := \pi_V \circ \tilde{T}$ and $\gamma' := \pi_V \circ \tilde{T}'$. Then $\Psi_\beta^{-i}(\gamma) = x_i + \pi_V \circ \tilde{T}_i$ and $\Psi_\beta^{-i}(\gamma') = x_i + \pi_V \circ \tilde{T}'_i$ with $x_i = -1/\beta^i \pi_V(v_i)$. Then $a(\gamma) \equiv a(\gamma') \pmod{\Gamma}$ and from $|\pi_V \circ \tilde{\Psi}_\beta^k(\tilde{T}_i)(t_{*,k-i}) - \pi_V \circ \tilde{\Psi}_\beta^k(\tilde{T}'_i)(t'_{*,k-i})| \leq B$ for all $k \in \mathbb{N}$, we see that $G(\gamma) = G(\gamma')$. Thus, $a(\gamma_T) \equiv a(\gamma) - G(\gamma) \equiv a(\gamma') - G(\gamma') \equiv a(\gamma_{T'}) \pmod{\Gamma}$, so $\pi(T) = \pi(T')$. It is clear from the definition of \sim_{gs} that $T \sim_{gs} T' \implies \Psi_\beta^k(T) \sim_{gs} \Psi_\beta^k(T')$ for all $k \in \mathbb{Z}$. Thus $\pi(\Psi_\beta^{-k}(T)) = \pi(\Psi_\beta^{-k}(T'))$ for all $k \geq 0$ and $\hat{\pi}(T) = \hat{\pi}(T')$. \square

Translated into the context of tilings-as-maps, the strong regional proximal relation reads as follows: $T \sim_{srp} T'$ provided for each $R > 0$ there are $S_R, S'_R \in \Omega_{\psi_\beta}$ and $t_R \in \mathbb{R}$ so that $T(t) = S_R(t)$, $T'(t) = S'_R(t)$, and $S_R(t - t_R) = S'_R(t - t_R)$ for all $|t| \leq R$.

Theorem 24. *If β is Pisot then the strong regional proximal and global shadowing relations on Ω_{ψ_β} are the same.*

This can be deduced (at least, in the unimodular case) as a corollary to Propositions 24 and 25 of [BG] which combine for a similar result in the higher dimensional setting. We give a simplified argument for the one-dimensional case at hand.

Let's say that strands γ, γ' corresponding to $T, T' \in \Omega_{\psi_\beta}$ are strong regionally proximal, and write $\gamma \sim_{srp} \gamma'$, provided there are: lifts \tilde{T}, \tilde{T}' of T, T' and $x \in V$ with $\gamma = x + \pi_V(\tilde{T}), \gamma' = x + \pi_V(\tilde{T}')$; and, for each $R > 0$, there are $t_R \in \mathbb{R}$ and $S_R, S'_R \in \Omega_{\psi_\beta}$ with lifts $\tilde{S}_R, \tilde{S}'_R$ so that $\tilde{T}(t) = \tilde{S}_R(t)$, $\tilde{T}'(t) = \tilde{S}'_R(t)$, $\tilde{S}_R(t - t_R) = \tilde{S}'_R(t - t_R)$, for all $|t| \leq R$.

Lemma 25. *There is $B \in \mathbb{R}$ so that if $\gamma \sim_{srp} \gamma'$, then $|\gamma(0) - \gamma'(0)| \leq B$.*

Proof. Let $n \in \mathbb{N}$ and $\rho < 1$ be so that $|(Lx)^s| \leq \rho$ for all $x \in X$ with $|x^s| \leq 1$. Let B_1 be such that if $j : [a, b] \rightarrow \mathcal{R}_{\psi_\beta}$ is any parameterization of a petal of \mathcal{R}_{ψ_β} with lift \tilde{j} , then $|\pi^s(\pi_V(\tilde{f}_{\psi_\beta}^n(\tilde{j}(t_1)))) - \pi^s(\pi_V(\tilde{f}_{\psi_\beta}^n(\tilde{j}(t_2))))| \leq B_1$ for all $t_1, t_2 \in [a, b]$. Now take B large enough so that $B/2 - \rho B/2 \geq B_1$. Then if γ is any strand corresponding to an element of Ω_{ψ_β} , $|\gamma^s(t_1) - \gamma^s(t_2)| \leq B/2$ for all $t_1, t_2 \in \mathbb{R}$ (since any compact piece of the image of γ is contained in a translation of some $\tilde{f}^{kn}(\tilde{j}([a, b]))$). Now if γ and γ' are strong regionally proximal, with S_1, S'_1 as in the above definition and $R = 1$, then $|(\gamma(0))^s - (\pi_V(\tilde{S}(t_1)))^s| \leq \frac{B}{2}$ and $|(\gamma'(0))^s - (\pi_V(\tilde{S}'(t_1)))^s| \leq \frac{B}{2}$. Since $\tilde{S}(t_1) = \tilde{S}'(t_1)$, we have $(\gamma(0))^u = (\gamma'(0))^u$ and $|(\gamma(0))^s - (\gamma'(0))^s| \leq B$. \square

Lemma 26. *If $\gamma \sim_{srp} \gamma'$ then $\Psi_\beta(\gamma) \sim_{srp} \Psi_\beta(\gamma')$ and $\Psi_\beta^{-1}(\gamma) \sim_{srp} \Psi_\beta^{-1}(\gamma')$.*

Proof. Suppose that γ and γ' correspond to $T, T' \in \Omega_{\psi_\beta}$ and $\gamma \sim_{srp} \gamma'$. Let $R > 0$ be given and let $S_{\beta R}, S'_{\beta R} \in \Omega_{\psi_\beta}$, $t_{\beta R} \in \mathbb{R}$, be as in the above definition of \sim_{srp} , with βR replacing R . We may then lift $\Psi_\beta^{-1}(T), \Psi_\beta^{-1}(S_{\beta R}), \Psi_\beta^{-1}(S'_{\beta R})$, and $\Psi_\beta^{-1}(T')$, successively, so that $\widetilde{\Psi_\beta^{-1}(T)}(t) = \widetilde{\Psi_\beta^{-1}(S_{\beta R})}(t)$, $\widetilde{\Psi_\beta^{-1}(S_{\beta R})}(t - (1/\beta)t_{\beta R}) = \widetilde{\Psi_\beta^{-1}(S_{\beta R})}(t - (1/\beta)t_{\beta R})$, and $\widetilde{\Psi_\beta^{-1}(T')}(t) = \widetilde{\Psi_\beta^{-1}(S'_{\beta R})}(t)$ for all $|t| \leq R$. Projecting to strands, and appropriately translating, shows that $\Psi_\beta^{-1}(\gamma) \sim_{srp} \Psi_\beta^{-1}(\gamma')$. The argument that $\Psi_\beta(\gamma) \sim_{srp} \Psi_\beta(\gamma')$ is similar. \square

Recall that $\sigma_i := \{te_i : 0 \leq t \leq 1\} \subset \mathbb{R}^{m+p}$, where e_i is the i -th standard unit vector. By the *segment of type i at $x \in \Gamma$* we will mean $(x + \pi_V(\sigma_i), i)$. Let $\Sigma := \{(\pi_V(y + \sigma_i), i) : y \in \mathbb{Z}^{m+p}, i = 1, \dots, m + p\}$ be the collection of all *segments*. If $\gamma = \pi_V \circ \tilde{T}$ is a strand with vertices in Γ , we may view the image of γ as a collection of segments. Let's say that *we can get from segment σ to segment σ' in one step* if there is a strand γ with σ and σ' in the image of γ , and let's say *we can get from σ to σ' in n steps* if there are segments $\sigma^0 = \sigma, \dots, \sigma^n = \sigma'$ so that we can get from σ^{i-1} to σ^i in one step for $i = 1, \dots, n$.

Ψ_β induces a map that takes segments to collections of segments: if σ is a segment defined by the image of $\pi_V \circ \tilde{T}$ on the interval $[t_1, t_2]$, then $\Psi_\beta(\sigma)$ is the collection of segments defined by the image of $\pi_V \circ \tilde{\Psi}_\beta(\tilde{T})$ on the interval $[\beta t_1, \beta t_2]$. This map has the property that if $\sigma' \in \Psi_\beta(\sigma)$ then $\Psi_\beta(\Sigma_\sigma) \subset \Sigma_{\sigma'}$.

Lemma 27. *If $\beta > 1$ is Pisot then there is $K \in \mathbb{N}$ so that for each $\sigma, \sigma' \in \cup_{k \geq K} \Psi_\beta^k(\Sigma)$ there is $n \in \mathbb{N}$ so that we can get from σ to σ' in n steps.*

Proof. Given a segment σ , let Σ_σ be the collection of all $\sigma' \in \Sigma$ for which there is $n \in \mathbb{N}$ such that we can get from σ to σ' in n steps. It is clear that:

- (1) the Σ_σ partition Σ ,
- (2) for each σ there are $\sigma', \sigma'' \in \Sigma_\sigma$ so that the terminal vertex of σ' is the initial vertex of σ and the initial vertex of σ'' is the terminal vertex of σ , and
- (3) if $\sigma' \in \Psi_\beta(\sigma)$, then $\Psi_\beta(\Sigma_\sigma) \subset \Sigma_{\sigma'}$.

Let's take $\sigma = (\pi_V(\sigma_1, 1))$ so that $\Psi_\beta(\Sigma_\sigma) \subset \Sigma_\sigma$. Note that if β is a simple Parry number ($m=0$), there is $K \in \mathbb{N}$ so that if $i \in \{1, \dots, p\}$ then $\sigma \in \Psi_\beta^k(\pi_V((\sigma_i, i)))$ for all $k \geq K$; and if β is non-simple, it follows from Lemma 12 that there is $K \in \mathbb{N}$ so that $\Psi_\beta^k(\pi_V((\sigma_i, i))) \cap \Psi_\beta^k(\pi_V((\sigma_j, j))) \neq \emptyset$ for all $i, j \in \{1, \dots, m + p\}$ and $k \geq K$. From this (and items (1)-(3) above) it follows that if

σ' and σ'' have a common vertex, then there is σ''' so that $\Psi_\beta^K(\Sigma_{\sigma'}) \cup \Psi_\beta^K(\Sigma_{\sigma''}) \subset \Sigma_{\sigma'''}$ with K as above. From connectedness of $spt(\Sigma) := \pi_V(\tilde{\mathcal{R}}_{\psi_\beta})$ we have $\Psi_\beta^k(\Sigma) \subset \Sigma_\sigma$ for all $k \geq K$. \square

Given the segment $\sigma = (x + \pi_V(\sigma_i), i)$, let's call $x + \pi_V(\sigma_i)$ the *support* of σ (denoted $spt(\sigma)$) and i its *type*. We'll say that the strand $\gamma = \pi_V \circ \tilde{T}$ *parameterizes* σ on $[s, s']$ if $\gamma([s, s']) = spt(\sigma)$ and $T([s, s'])$ is the i -th loop of \mathcal{R}_{ψ_β} . Note that if γ parameterizes σ on $[s, s']$, then $\gamma - t$ parameterizes σ on $[s - t, s' - t]$. Suppose now that we can get from σ to σ' in n steps. There are then segments $\sigma = \sigma^0, \dots, \sigma^n = \sigma'$ and strands $\gamma^1, \dots, \gamma^n$ so that (on some intervals) γ^i parameterizes both σ^{i-1} and σ^i for $i = 1, \dots, n$. By replacing γ^i by $\gamma^i - t_i$, with the appropriately chosen t_i for $i = 2, \dots, n$, we may arrange that: γ^1 parameterizes σ^0 on $[s_0, s'_0]$ and σ^1 on $[s_1, s'_1]$; γ^2 parameterizes σ^1 on $[s_1, s'_1]$ and σ^2 on $[s_2, s'_2]$; ... ; and γ^n parameterizes σ^{n-1} on $[s_{n-1}, s'_{n-1}]$ and σ_n on $[s_n, s'_n]$. Then if $\gamma^i = x_i + \pi_V \circ \tilde{S}_i$, the tilings S_i have the property: $S_i(t) = S_{i+1}(t)$ for $t \in [s_i, s'_i]$ and $i = 1, \dots, n-1$.

Lemma 28. *Suppose that $T, T' \in \Omega_{\psi_\beta}$ and there is $n \in \mathbb{N}$ so that, for each $R > 0$, there are $S_i = S_i(R) \in \Omega_{\psi_\beta}$, $i = 1, \dots, n$, and $t_i = t_i(R) \in \mathbb{R}$, $i = 1, \dots, n+1$, so that for all $|t| < R$: $(T - t_1)(t) = (S_1 - t_1)(t)$; $(S_i - t_{i+1})(t) = S_{i+1} - t_{i+1}(t)$, $i = 1, \dots, n-1$; and $(S_n - t_{n+1})(t) = (T' - t_{n+1})(t)$. Then $T \sim_{srp} T'$.*

Proof. Let $\epsilon > 0$ be given. There is then $R > 0$ so that if $S, S' \in \Omega_{\psi_\beta}$ are such that $S(t) = S'(t)$ for $|t| < R$, then $d(\pi_{max}(S), \pi_{max}(S')) < \epsilon/(n+1)$. Since \mathbb{R} acts by isometries on $\hat{\mathbb{T}}_\beta^d$ and $\pi_{max}(S - t) = \pi_{max}(S) - t$, we have, for such S, S' , that $d(\pi_{max}(S - t), \pi_{max}(S' - t)) < \epsilon/(n+1)$ for all $t \in \mathbb{R}$. For this R , let the $S_i(R)$ and $t_i(R)$ be as hypothesized. Then: $d(\pi_{max}(T), \pi_{max}(S_1)) < \epsilon/(n+1)$; $d(\pi_{max}(S_i), \pi_{max}(S_{i+1})) < \epsilon/(n+1)$, $i = 1, \dots, n-1$; and $d(\pi_{max}(S_n), \pi_{max}(T')) < \epsilon/(n+1)$. Hence $d(\pi_{max}(T), \pi_{max}(T')) < \epsilon$, so $\pi_{max}(T) = \pi_{max}(T')$ and $T \sim_{srp} T'$. \square

Proof. (Of Theorem 24) Suppose that $T, T' \in \Omega_{\psi_\beta}$ with $T \sim_{rp} T'$. There are then corresponding strands γ, γ' that are strong regionally proximal. By Lemma 26, $\Psi_\beta^k(\gamma) \sim_{srp} \Psi_\beta^k(\gamma')$ for all $k \in \mathbb{Z}$ and by so Lemma 25, $T \sim_{gs} T'$.

Now suppose that $T \sim_{gs} T'$. Let $B < \infty$, $\tilde{T}, \tilde{T}', \tilde{T}_i, \tilde{T}'_i$, and $v_i \in \Gamma$ be as in Lemma 22.

Recall that, for $S \in \Omega_{\psi_\beta}$, $t_*(S) = \sup\{t \leq 0 : S(t) = *\}$. Let's set $t_1(S) := \inf\{t > t_0(S) : S(t) = *\}$. As in Lemma 22 we let $t_{*,j} = t_*(\Psi_\beta^j(T))$ and $t'_{*,j} = t_*(\Psi_\beta^j(T'))$; correspondingly, we set $t_{1,j} = t_1(\Psi_\beta^j(T))$ and $t'_{1,j} = t_1(\Psi_\beta^j(T'))$. For each $(k, i) \in \mathbb{N}^2$ let $\gamma_{(k,i)}$ and $\gamma'_{(k,i)}$ be the strands:

$$\gamma_{(k,i)} := \pi_V \circ \tilde{\Psi}_\beta^k(\tilde{T}_i)$$

and

$$\gamma'_{(k,i)} := \pi_V \circ \tilde{\Psi}_\beta^k(\tilde{T}'_i).$$

Then $\gamma_{(k,i)}$ parameterizes a segment, call it $\sigma_{(k,i)}$, on $[t_{*,k-i}, t_{1,k-i}]$ and $\gamma'_{(k,i)}$ parameterizes a segment $\sigma'_{(k,i)}$ on $[t'_{*,k-i}, t'_{1,k-i}]$. We observe:

- (1) The pair of segments $(\sigma_{(k,i)}, \sigma'_{(k,i)})$ depends, up to translation by an element of Γ , only on $k - i$.
- (2) $\sigma_{(k,i)}, \sigma'_{(k,i)} \in \Psi_\beta^k(\Sigma)$, so if K is as in Lemma 27 and $k \geq K$, then there is n so that we can get from $\sigma_{(k,i)}$ to $\sigma'_{(k,i)}$ in n steps.

From (1) it follows that there are $K \leq i_1 < i_2 < i_3 \cdots$ so that the pairs $(\sigma_{(K,i_j)}, \sigma'_{(K,i_j)})$ are all the same, up to translation by elements of Γ , and then, by (2), there is n , independent of j , so that we can get from $\sigma_{(K,i_j)}$ to $\sigma'_{(K,i_j)}$ in n steps.

Fix $j \geq 1$ and let $\sigma^0 = \sigma_{(K,i_j)}, \sigma^1, \dots, \sigma_n = \sigma'_{(K,i_j)}$ be such that we can get from σ^{i-1} to σ^i in one step for $i = 1, \dots, n$. There are then tilings S_i , $i = 1, \dots, n$, and intervals $[s_i, s'_i]$, $i = 0, \dots, n$, with $[s_0, s'_0] = [t_{*,K-i_j}, t_{1,K-i_j}]$, and lifts \tilde{S}_i so that the strand $\pi_V \circ \tilde{S}_i$ parameterizes σ^{i-1} on $[s_{i-1}, s'_{i-1}]$ and σ^i on $[s_i, s'_i]$, $i = 1, \dots, n$.

Claim: $[s_n, s'_n] = [t'_{*,K-i_j}, t'_{1,K-i_j}]$.

To see this, note that for any strand γ , we have: $\pi^u(\gamma(t)) = \pi^u(\gamma(0)) + t\omega$ for all $t \in \mathbb{R}$. Thus, if γ and γ' are strands for which $\pi^u(\gamma(t)) = \pi^u(\gamma'(t))$ for some t then $\pi^u(\gamma(t)) = \pi^u(\gamma'(t))$ for all t . Hence, $\pi^u(\pi_V \circ \tilde{\Psi}_\beta^K(\tilde{T}_{i_j})(t)) = \pi^u(\pi_V \circ \tilde{S}_i(t))$ for all $t \in \mathbb{R}$ and $i = 1, \dots, n$. We also know that $\pi_V \circ \tilde{\Psi}_\beta^K(\tilde{T}_{i_j})(0) - \pi_V \circ \tilde{\Psi}_\beta^K(\tilde{T}'_{i_j})(0) \in E^s$ (if γ and γ' are such that $\gamma(0) - \gamma'(0) \notin E^s$, then $|\Psi_\beta^n(\gamma)(0) - \Psi_\beta^n(\gamma')(0)|$ grows without bound as $n \rightarrow \infty$, and then so also does $|a_n(\gamma) - a_n(\gamma')|$). Thus $\pi^u(\pi_V \circ \tilde{\Psi}_\beta^K(\tilde{T}_{i_j})(t)) = \pi^u(\pi_V \circ \tilde{\Psi}_\beta^K(\tilde{T}'_{i_j})(t))$ for all t . Now $\pi_V \circ \tilde{\Psi}_\beta^K(\tilde{T}'_{i_j})$ and $\pi_V \circ \tilde{S}_n$ parameterize the same segment $\sigma'_{(K,i_j)} = \sigma_n$ on the intervals $[t'_{*,K-i_j}, t'_{1,K-i_j}]$ and $[s_n, s'_n]$, resp., while $\pi^u(\pi_V \circ \tilde{\Psi}_\beta^K(\tilde{T}'_{i_j})(t)) = \pi^u(\pi_V \circ \tilde{S}_n(t))$ for all t , so these intervals must be the same, as claimed.

If $r = \min\{\text{length}(\tau_i) : i = 1, \dots, m+p\}$, then each of the intervals $[s_i, s'_i]$ occurring above has length at least r . Given $R > 0$, let j be large enough so that $\beta^{i_j-K}r > 2R$. The hypotheses of Lemma 28 are then satisfied, with $\Psi_\beta^{i_j-K}(S_i)$ playing the role of S_i in the lemma, and $t_i(R) := (\frac{s_{i-1}+s'_{i-1}}{2})\beta^{i_j-K}$. \square

Corollary 29. $\hat{\pi} = \pi_{\max} : \Omega_{\psi_\beta} \rightarrow \hat{\mathbb{T}}^d \simeq \hat{\mathbb{T}}_\beta^d$.

To connect the foregoing with arithetical coding, let $\Pi : X_\beta^+ \rightarrow I$ by $\Pi((x_i)) := \sum_{i=1}^\infty x_i \beta^{-i}$. Then $\Pi \circ \sigma = T_\beta \circ \Pi$, Π is a.e. one-to-one, and so is $\hat{\Pi} : x_\beta \rightarrow \varprojlim T_\beta$. (The unique measure η of maximal entropy for T_β is equivalent to Lebesgue measure ([Hof1, Hof2]), for $\varprojlim T_\beta$ we use the measure induced by η .) The map from Ω_{ψ_β} to $[0, 1)$ given by $T \mapsto -t_*(T)$ is surjective and from $-t_*(\Psi_\beta(T)) = T_\beta(-t_*(T))$, there is an induced map $\widehat{-t_*} : \Omega_{\psi_\beta} \rightarrow \varprojlim T_\beta$. The image of $\widehat{-t_*}$ is $\varprojlim T_\beta|_{[0,1]}$, which is a full measure subset of $\varprojlim T_\beta$. For $T \in \Omega_{\psi_\beta}$ and $i \in \mathbb{Z}$, let $x_i(T) := \lfloor -\beta t_*(\Psi_\beta^i(T)) \rfloor$. Then $(x_i(T)) \in X_\beta$ and $\widehat{-t_*}(T) = \hat{\Pi}((x_i(T)))$. It follows that $\{x_i(T) : T \in \Omega_{\psi_\beta}\}$ has full measure in X_β .

Recall that ω is the positive right eigenvector of L , normalized with $\langle l^t, \omega \rangle = 1$, $l = (l_1, \dots, l_{m+p})$ being the left eigenvector of the substitution matrix A with $l_i = \text{length}(\tau_i)$ (note that $\sum_{i=1}^{m+p} l_i = 1$). Then $\pi^u(\pi_V(e_i)) = l_i \omega$ for each $i \in \{1, \dots, m+p\}$ and since ω is ‘totally irrational’ in V , $\pi^u : \Gamma \rightarrow E^u$ is one-to-one and we have $\Gamma^u := \pi^u(\Gamma) = \langle l_1 \omega, \dots, l_{m+p} \omega \rangle_{\mathbb{Z}} = \langle l_1, \dots, l_{m+p} \rangle_{\mathbb{Z}} \omega$. We claim that $\langle l_1, \dots, l_{m+p} \rangle_{\mathbb{Z}} = \mathbb{Z}[\beta]$. To see this, recall the notation $z^j = T_\beta^{j-1}(1)$. Clearly, $z^j \in \mathbb{Z}[\beta]$. Since the endpoints of the τ_i are contained in $\{0\} \cup \{z^j : j = 1, \dots, m+p\}$, we have $\langle l_1, \dots, l_{m+p} \rangle_{\mathbb{Z}} \subset \mathbb{Z}[\beta]$. On the other hand, for each $j = 1, \dots, d \leq m+p$, there is $i \in \{1, \dots, m+p\}$ with $\max(\tau_i) = z^j$. We have $z^1 = 1$ and, for $j \geq 2$, $z^j = \beta^{j-1} + q_{j-2}(\beta)$, where $q_{j-2}(x) \in \mathbb{Z}[x]$ has degree $j-2$. Then $\beta^j = z^{j+1} - q_{j-1}(\beta) = l_1 + l_2 + \dots + l_{j+1} - q_{j-1}(\beta) \in \langle l_1, \dots, l_{m+p} \rangle_{\mathbb{Z}}$ for all j , inductively. We thus have $\Gamma^u = \mathbb{Z}[\beta]\omega$.

Now let $e := -\pi_V(\sum_{i=1}^{m+p} e_i) \in \Gamma$. From $\pi^u \circ L = \beta\pi^u$ and the above claim, we see that $e, Le, \dots, L^{d-1}e$ generates Γ , that is, $\bar{y} := y + \Gamma$, $y := e^u$, is a fundamental homoclinic point for $F_L : V/\Gamma \rightarrow V/\Gamma$.

For each $i \in \mathbb{Z}$ and $T \in \Omega_{\psi_\beta}$ let $p_i(T) := p(\Psi_\beta^i(T)) = \gamma(t_*(\Psi_\beta^i(T))) - \gamma(\beta t_*(\Psi_\beta^{i-1}(T)))$, γ any strand corresponding to $\Psi_\beta^i(T)$, be the abelian prefix of $\Psi_\beta^i(T)$ and let $a_i := a_i(\gamma_T)$. Then (a_i) is bounded and $a_i = p_i + La_{i-1}$ for all $i \in \mathbb{Z}$. We have $a_0 = p_0 + La_{-1} = p_0 + Lp_{-1} + L^2a_{-2} = \dots$. Since the a_i are bounded,

$$a_0^s = \sum_{i=0}^{\infty} L^i p_{-i}^s,$$

Similarly,

$$a_0^u = - \sum_{i=1}^{\infty} L^{-i} p_i^u.$$

Since $p_i(T) \in \Gamma$, $L^i p_{-i}^s \equiv -L^i p_{-i}^u \pmod{\Gamma}$ for $i \geq 0$, and we have

$$a_0 = a_0(T) \equiv \sum_{i=-\infty}^{\infty} L^i (-p_{-i}^u(T)), \pmod{\Gamma}.$$

Thus, since $p_i^u(T) = x_i(T)\omega$,

$$\pi(T) = a_0(\gamma_T) + \Gamma = - \sum_{i=-\infty}^{\infty} (x_i(T)\beta^{-i}\omega + \Gamma).$$

On the other hand, with fundamental homoclinic point $\bar{y} = e^u = -\omega + \Gamma$ (and $\hat{y} = (-\omega + \Gamma, -\beta^{-1}\omega + \Gamma, \dots)$), we have $h_{\bar{y}}((x_i(T))) = \sum_{i=-\infty}^{\infty} x_i(T)\hat{F}_L^{-i}(-\omega + \Gamma, -1/\beta\omega + \Gamma, \dots) = -(\sum_{i=-\infty}^{\infty} (x_i(T)\beta^{-i}\omega + \Gamma), \sum_{i=-\infty}^{\infty} (\beta^{-i-1}\omega + \Gamma), \dots)$. Thus we see that

$$\hat{\pi}(T) = h_{\bar{y}}((x_i(T)))$$

for all $T \in \Omega_{\psi_\beta}$.

From Theorem 15 and Corollary 29, $h_{\bar{y}}$, restricted to $\{(x_i(T)) : T \in \Omega_{\psi_\beta}\}$ is a.e one-to-one. Since $\{(x_i(T)) : T \in \Omega_{\psi_\beta}\}$ has full measure in X_β , we have proved Corollary 18.

We finish with an argument that a stronger formulation than Property (W) is equivalent to pure discrete spectrum of $(\Omega_{\psi_\beta}, \mathbb{R})$. For $T \in \Omega_{\psi_\beta}$ we write $W^s(T)$ for $\{T' \in \Omega_{\psi_\beta} : d(\Psi_\beta^n(T'), \Psi_\beta(T)) \rightarrow 0 \text{ as } n \rightarrow \infty\}$ and for $\hat{z} \in \hat{\mathbb{T}}_\beta^d$, $W^s(\hat{z}) := \{\hat{z}' : d(\hat{F}_M^n(\hat{z}'), \hat{F}_M^n(\hat{z})) \rightarrow 0 \text{ as } n \rightarrow \infty\}$. (Here $\hat{F}_M : \hat{\mathbb{T}}_\beta^d \rightarrow \hat{\mathbb{T}}_\beta^d$ is the hyperbolic automorphism induced on the maximal equicontinuous factor of Ω_{ψ_β} .)

Lemma 30. *If $(\Omega_{\psi_\beta}, \mathbb{R})$ has pure discrete spectrum and $T, T' \in \Omega_{\psi_\beta}$ are such that $T \sim_s T'$, then $\{t : T - t \sim_s T' - t\}$ is open and dense in \mathbb{R}*

Proof. We have previously observed that $T - t \sim_s T' - t$ is an open property. If $T \sim_s T'$ then $d(\pi_{\max}(\Psi_\beta^k(T - t)), \pi_{\max}(\Psi_\beta^k(T' - t))) = d(\hat{F}_M^k((\pi_{\max}(T)) - \beta^k t), \hat{F}_M^k((\pi_{\max}(T')) - \beta^k t)) \rightarrow 0$ as $k \rightarrow \infty$ uniformly in t . Suppose there is an interval $J = (t_0 - \epsilon, t_0 + \epsilon)$ with $T - t \not\sim_s T' - t$ for all $t \in J$. Take $k_i \rightarrow \infty$ so that $\Psi_\beta^{k_i}(T - t_0) \rightarrow S \in \Omega_{\psi_\beta}$ and $\Psi_\beta^{k_i}(T' - t_0) \rightarrow S' \in \Omega_{\psi_\beta}$. Then $S \sim_{srp} S'$ (since $d(\pi_{\max}(S), \pi_{\max}(S')) = 0$) and $S \cap S' = \emptyset$ (if $B_0[S - t] = B_0[S' - t]$ for some t then $B_0[\Psi_\beta^{k_i}(T - (t_0 - t/\beta^{k_i}))] = B_0[\Psi_\beta^{k_i}(T' - (t_0 - t/\beta^{k_i}))]$ for all large i , contradicting

$T - t' \sim_s T' - t'$ with $t' = t_0 - t/\beta^{k_i} \in J$). But then the coincidence rank of ψ_β is at least two, so $(\Omega_{\psi_\beta}, \mathbb{R})$ does not have pure discrete spectrum. \square

A few observations:

- (1) For any $T, T' \in \Omega_{\psi_\beta}$, $\{t : T' - t \in W^s(T)\}$ is dense in \mathbb{R} . (This is a consequence of the primitivity of ψ_β .)
- (2) $d(t\beta^k\omega, \Gamma) \rightarrow 0$ as $k \rightarrow \infty$ if and only if $t \in \mathbb{Z}[1/\beta]$. Hence, if $\hat{0} = (\bar{0}, \bar{0}, \dots)$, $\hat{0} - t \in W^s(\hat{0})$ if and only if $t \in \mathbb{Z}[1/\beta]$.
- (3) $-t_*(\Psi_\beta(T)) = T_\beta(-t_*(T))$ for all $T \in \Omega_{\psi_\beta}$.
- (4) Since $\pi_{max}^{-1}(\hat{0})$ is finite ([BBK], or [BK]) and Ψ_β -invariant there is $N > 0$ so that $\Psi_\beta^N(T) = T$ for all $T \in \pi_{max}^{-1}(\hat{0})$.
- (5) Given $T, T' \in \Omega_{\psi_\beta}$ with $T' \in W^s(T)$, there is $\epsilon > 0$ so that $T' - t \in W^s(T - t)$ for all $|t| < \epsilon$. (This is a consequence of ‘local product structure’ - see [AP]. Or, invoke the openness of the relation \sim_s , which is proved in [BO].)

Proposition 31. *If β is Pisot, then $(\Omega_{\psi_\beta}, \mathbb{R})$ has pure discrete spectrum if and only if for all $t \in \mathbb{Z}[1/\beta] \cap \mathbb{R}^+$ the set $\{t' \in Fin(\beta) : t + t' \in Fin(\beta)\}$ is dense in \mathbb{R}^+ .*

Proof. Suppose that $(\Omega_{\psi_\beta}, \mathbb{R})$ has pure discrete spectrum and pick $t \in \mathbb{Z}[1/\beta] \cap \mathbb{R}^+$. For given $i \in \{1, \dots, p\}$, there is $T \in \pi_{max}^{-1}(\hat{0})$ so that $T_0^i - t \in W^s(T)$ (this follows from item (2)). Now, $\{t' > 0 : T_0^i - t' \in W^s(T_0^i)\}$ is dense in \mathbb{R}^+ (from (1)) and $\{t' \in \mathbb{R} : T_0^i - t' \sim_s T - t'\}$ is open and dense in \mathbb{R} (Theorem 9). Then $C := \{t' > 0 : T - t', T_0^i - t' \in W^s(T_0^i)\}$ is dense in \mathbb{R}^+ and all these t' are also in $Fin(\beta)$. Indeed, if $t \geq 0$ and $T_0^i - t' \in W^s(T_0^i)$, let k be large enough so that $t'/\beta^{kp} < 1$. Then $T_\beta^{kp+n}(t'/\beta^{kp}) = T_\beta^{kp+n}(-t_*(T_0^i - t'/\beta^{kp})) = -t_*(\Psi_\beta^n(T_0^i - t')) = 0$ for n large enough that $\tau_1 \in \Psi_\beta^n(T_0^i - t')$, so t'/β^{kp} , and hence t' , is in $Fin(\beta)$. We have $T_0^i - t \sim_s T$, so by Lemma 30, the set $D := \{t' : T_0^i - t - t' \sim_s T - t'\}$ is open and dense in \mathbb{R} . Now $C \cap D$ is dense in \mathbb{R}^+ and for $t' \in C \cap D$ we have $T_0^i - t - t' \sim_s T - t' \sim_s T_0^i$, so $t + t' \in Fin(\beta)$ (as above). Thus if $t' \in C \cap D$ then t' and $t + t'$ are both in $Fin(\beta)$.

Now suppose that $F(t) := \{t' \in Fin(\beta) : t + t' \in Fin(\beta)\}$ is dense in \mathbb{R}^+ for all $t \in \mathbb{Z}[1/\beta] \cap \mathbb{R}^+$ and let $T \in \pi_{max}^{-1}(\hat{0})$. Fix $i \in \{1, \dots, p\}$ and let $t \in \mathbb{R}^+$ be so that $T_0^i - t \sim_s T$. Then $t \in \mathbb{Z}[1/\beta]$. Let $\epsilon > 0$ be small enough so that if $|t'| < \epsilon$, then $T_0^i - t - t' \sim_s T - t'$. Note that if $t' \in Fin(\beta)$, then $t'/\beta^{kp} \in Fin(\beta)$ and we have $0 = T_\beta^{n+kp}(t'/\beta^{kp}) = T_\beta^{n+kp}(-t_*(T_0^i - t'/\beta^{kp})) = -t_*(\Psi_\beta^n(T_0^i - t'))$ for large n and k big enough so that $t'/\beta^{kp} \in [0, 1)$. Thus, $T_0^i - t' \in W^s(T_0^j)$ for some $j \in \{1, \dots, p\}$. Then for $t' \in F(t) \cap [0, \epsilon)$ we have $T_0^i - t' \sim_s T_0^j$, $T_0^i - (t + t') \sim_s T - t'$ and $T_0^i - (t + t') \sim_s T_0^k$ for some $j, k \in \{1, \dots, p\}$. Since $T_0^j - t'' \sim_s T_0^k - t''$ for all $t'' > 0$, we see that $T_0^i \sim_s T$ densely on $[0, \epsilon)$. Using the Ψ_β -periodicity of T_0^i and T , this means that $W(T) := \{t' > 0 : T_0^i - t' \sim T - t'\}$ is open and dense in \mathbb{R}^+ . Thus, for each $T, T' \in \pi_{max}^{-1}(\hat{0})$, $W(T) \cap W(T') \neq \emptyset$, and then $T \cap T' \neq \emptyset$. Thus the coincidence rank of ψ_β is 1 and $(\Omega_{\psi_\beta}, \mathbb{R})$ has pure discrete spectrum. \square

It follows from Theorem 15 and Proposition 31 that all Pisot β are weakly finitary, proving Corollary 17.

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